Quantum Field Theory: Problem Solutions

Mark Srednicki
University of California, Santa Barbara
mark@physics.ucsb.edu

1 Attempts at relativistic quantum mechanics

1.1) $\beta^2 = 1 \Rightarrow$ eigenvalue-squared $= 1 \Rightarrow$ eigenvalue $= \pm 1$. $\alpha_1^2 = 1 \Rightarrow \operatorname{Tr}\beta = \operatorname{Tr}\alpha_1^2\beta$. Cyclic property of the trace $\Rightarrow \operatorname{Tr}\alpha_1^2\beta = \operatorname{Tr}\alpha_1\beta\alpha_1$. Then $\{\alpha_1, \beta\} = 0 \Rightarrow \operatorname{Tr}\alpha_1\beta\alpha_1 = -\operatorname{Tr}\alpha_1^2\beta = -\operatorname{Tr}\beta$. Thus $\operatorname{Tr}\beta$ equals minus itself, and so must be zero. $\operatorname{Tr}\alpha_i = 0$ follows from this analysis by taking $\beta \to \alpha_i$ and $\alpha_1 \to \beta$.

1.2) For notational simplicity, switch to a discrete notation:

$$\begin{aligned} \int &= \int d^3x_1 \ldots d^3x_n \ , \\ \delta_{xy} &= \delta^3(\mathbf{x}-\mathbf{y}) \ , \\ a_1 &= a(\mathbf{x}_1) \ , \\ \psi &= \psi(\mathbf{x}_1, \ldots, \mathbf{x}_n; t) \ . \end{aligned} \tag{1.40}$$

Using

$$[X, AB\ldots C] = [X,A]B\ldots C + A[X,B]\ldots C + \ldots + AB\ldots[X,C] \ , \tag{1.41}$$

which follows from writing out the terms on both sides, we have

$$\begin{aligned} [a_x^\dagger a_y, a_1^\dagger \ldots a_n^\dagger] = {} & [a_x^\dagger a_y, a_1^\dagger]a_2^\dagger \ldots a_n^\dagger + a_1^\dagger[a_x^\dagger a_y, a_2^\dagger]a_3^\dagger \ldots a_n^\dagger \\ & + \ldots + a_1^\dagger \ldots a_{n-1}^\dagger[a_x^\dagger a_y, a_n^\dagger] \ . \end{aligned} \tag{1.42}$$

We have

$$\begin{aligned} [a_x^\dagger a_y, a_i^\dagger] &= a_x^\dagger[a_y, a_i^\dagger]_\mp \pm [a_x^\dagger, a_i^\dagger]_\mp a_y \ , \\ &= \delta_{iy} a_x^\dagger \end{aligned} \tag{1.43}$$

where $[A,B]_\mp = AB \mp BA$. Using this and $a_y|0\rangle = 0$, we find

$$(a_x^\dagger a_y)a_1^\dagger \ldots a_n^\dagger|0\rangle = \sum_{i=1}^n (a_1^\dagger \ldots a_n^\dagger)_{i\to x}\delta_{iy}|0\rangle \ . \tag{1.44}$$

Similarly, we have

$$(a_x^\dagger a_y^\dagger a_y a_x)a_1^\dagger \ldots a_n^\dagger|0\rangle = \sum_{i,j=1}^n (a_1^\dagger \ldots a_n^\dagger)_{\substack{i\to x \\ j\to y}}|0\rangle \tag{1.45}$$

for both bosons and fermions. (Extra minus signs with fermions cancel when we move $a_x^\dagger$ and $a_y^\dagger$ into the positions formerly occupied by $a_i^\dagger$ and $a_j^\dagger$.)

Now consider

$$\int d^3x \ a^\dagger(\mathbf{x})\nabla_x^2 a(\mathbf{x})|\psi\rangle = \sum_{i=1}^n \int d^3x \int (\psi\nabla_x^2\delta_{xi})(a_1^\dagger \ldots a_n^\dagger)_{i\to x}|0\rangle \ . \tag{1.46}$$

We have $\nabla_x^2\delta_{xi} = \nabla_i^2\delta_{xi}$. Then we can integrate by parts to put ∇_i^2 onto ψ, and then integrate over x using the delta function to get

$$\int d^3x \ a^\dagger(\mathbf{x})\nabla_x^2 a(\mathbf{x})|\psi\rangle = \sum_{i=1}^n \int (\nabla_i^2\psi)a_1^\dagger \ldots a_n^\dagger|0\rangle \ . \tag{1.47}$$

Similarly,

$$\int d^3x\, U(\mathbf{x})a^\dagger(\mathbf{x})a(\mathbf{x})|\psi\rangle = \sum_{i=1}^{n} \int U(\mathbf{x}_i)|\psi\rangle \ , \tag{1.48}$$

and

$$\int d^3x\, d^3y\, V(\mathbf{x}-\mathbf{y})a^\dagger(\mathbf{x})a^\dagger(\mathbf{y})a(\mathbf{y})a(\mathbf{x})|\psi\rangle = \sum_{i,j=1}^{n} V(\mathbf{x}_i-\mathbf{x}_j)|\psi\rangle \ , \tag{1.49}$$

which yields the desired result.

1.3) $N = \sum_i a_i^\dagger a_i$. Then $[N, a_j^\dagger] = +a_j^\dagger$ and $[N, a_j] = -a_j$ for both bosons and fermions. Thus, using eq. (1.41), we find

$$[N, a_{i_1}^\dagger \ldots a_{i_n}^\dagger a_{j_1} \ldots a_{j_m}] = (n-m)a_{i_1}^\dagger \ldots a_{i_n}^\dagger a_{j_1} \ldots a_{j_m} \ . \tag{1.50}$$

Thus if the number of a's equals the number of $a^\dagger$'s, the operator commutes with N.

2 Lorentz Invariance

2.1) Start with eq. (2.3) and let $\Lambda^\mu{}_\rho + \delta^\mu{}_\rho + \delta\omega^\mu{}_\rho$. We will always drop terms that are $O(\delta\omega^2)$ or higher. Then we have

$$\begin{aligned} g_{\rho\sigma} &= g_{\mu\nu}(\delta^\mu{}_\rho + \delta\omega^\mu{}_\rho)(\delta^\nu{}_\sigma + \delta\omega^\nu{}_\sigma) \\ &= g_{\mu\nu}(\delta^\mu{}_\rho\delta^\nu{}_\sigma + \delta^\mu{}_\rho\delta\omega^\nu{}_\sigma + \delta\omega^\mu{}_\rho\delta^\nu{}_\sigma) \\ &= g_{\rho\sigma} + g_{\rho\nu}\delta\omega^\nu{}_\sigma + g_{\mu\sigma}\delta\omega^\mu{}_\rho \\ &= g_{\rho\sigma} + \delta\omega_{\rho\sigma} + \delta\omega_{\sigma\rho}\ , \end{aligned} \tag{2.39}$$

which implies $\delta\omega_{\rho\sigma} + \delta\omega_{\sigma\rho} = 0$.

2.2) Let $\Lambda' = 1 + \delta\omega$, so that $U(\Lambda') = I + \frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}$. Then we have

$$U(\Lambda)^{-1}(I + \tfrac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu})U(\Lambda) = I + \tfrac{i}{2\hbar}\delta\omega_{\mu\nu}U(\Lambda)^{-1}M^{\mu\nu}U(\Lambda)\ . \tag{2.40}$$

Using $\Lambda^{-1}(1{+}\delta\omega)\Lambda = 1 + \Lambda^{-1}\delta\omega\Lambda$, we also have

$$U(\Lambda^{-1}(1{+}\delta\omega)\Lambda) = I + \tfrac{i}{2\hbar}(\Lambda^{-1}\delta\omega\Lambda)_{\rho\sigma}M^{\rho\sigma}\ . \tag{2.41}$$

Now we use

$$\begin{aligned} (\Lambda^{-1}\delta\omega\Lambda)_{\rho\sigma} &= (\Lambda^{-1})_\rho{}^\mu\delta\omega_{\mu\nu}\Lambda^\nu{}_\sigma \\ &= \Lambda^\mu{}_\rho\delta\omega_{\mu\nu}\Lambda^\nu{}_\sigma \\ &= \delta\omega_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma\ . \end{aligned} \tag{2.42}$$

Equating the right-hand sides of eqs. (2.40) and (2.41) gives eq. (2.12), which, by the argument in the text, then yields eq. (2.13).

2.3) We start with

$$U(1{+}\delta\omega)^{-1} = U(1{-}\delta\omega) = I - \tfrac{i}{2\hbar}\delta\omega_{\rho\sigma}M^{\rho\sigma}\ , \tag{2.43}$$

and so, for any operator A,

$$U(1{+}\delta\omega)^{-1}AU(1{+}\delta\omega) = A + \tfrac{i}{2\hbar}\delta\omega_{\rho\sigma}[A, M^{\rho\sigma}]\ . \tag{2.44}$$

In particular,

$$U(1{+}\delta\omega)^{-1}M^{\mu\nu}U(1{+}\delta\omega) = M^{\mu\nu} + \tfrac{i}{2\hbar}\delta\omega_{\rho\sigma}[M^{\mu\nu}, M^{\rho\sigma}]\ . \tag{2.45}$$

Also,

$$\begin{aligned} &(1{+}\delta\omega)^\mu{}_\rho(1{+}\delta\omega)^\nu{}_\sigma M^{\rho\sigma} \\ &\quad= M^{\mu\nu} + \delta\omega^\mu{}_\rho M^{\rho\nu} + \delta\omega^\nu{}_\sigma M^{\mu\sigma} \\ &\quad= M^{\mu\nu} + \delta\omega_{\sigma\rho}g^{\sigma\mu}M^{\rho\nu} + \delta\omega_{\rho\sigma}g^{\rho\nu}M^{\mu\sigma} \\ &\quad= M^{\mu\nu} - \delta\omega_{\rho\sigma}(g^{\sigma\mu}M^{\rho\nu} - g^{\rho\nu}M^{\mu\sigma}) \\ &\quad= M^{\mu\nu} - \tfrac{1}{2}\delta\omega_{\rho\sigma}(g^{\sigma\mu}M^{\rho\nu} - g^{\rho\nu}M^{\mu\sigma} - g^{\rho\mu}M^{\sigma\nu} + g^{\sigma\nu}M^{\mu\rho})\ . \end{aligned} \tag{2.46}$$

Equating the right-hand sides of eqs. (2.45) and (2.46) and matching the coefficients of $\delta\omega_{\rho\sigma}$ gives

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\hbar(g^{\sigma\mu}M^{\rho\nu} - g^{\rho\nu}M^{\mu\sigma} - g^{\rho\mu}M^{\sigma\nu} + g^{\sigma\nu}M^{\mu\rho})\ . \tag{2.47}$$

Using $g^{\mu\nu} = g^{\nu\mu}$ and $M^{\mu\nu} = -M^{\nu\mu}$ then yields eq. (2.16).

2.4) From eq. (2.47), we have

$$\begin{aligned}[M^{i0},M^{j0}] &= i\hbar(g^{0i}M^{j0} - g^{j0}M^{i0} - g^{ji}M^{00} + g^{00}M^{ij}) \\ &= i\hbar(0 - 0 - 0 + (-1)M^{ij})\ . \end{aligned} \tag{2.48}$$

Using $M^{00}=0$, $M^{i0}=K^i$, and $M^{ij}=\varepsilon^{ijk}J^k$, eq. (2.48) becomes

$$[K^i,K^j] = -i\hbar\varepsilon^{ijk}J^k\ . \tag{2.49}$$

Similarly,

$$\begin{aligned}[M^{kl},M^{j0}] &= i\hbar(g^{0k}M^{jl} - g^{jl}M^{k0} - g^{jk}M^{0l} + g^{0k}M^{lj}) \\ &= i\hbar(0 - \delta^{jl}K^k - \delta^{jk}(-K^l) + 0)\ . \end{aligned} \tag{2.50}$$

Multiply by $\frac{1}{2}\varepsilon^{ikl}$ and use $J^i = \frac{1}{2}\varepsilon^{ikl}M^{kl}$ to get

$$\begin{aligned}[J^i,K^j] &= \tfrac{1}{2}i\hbar(-\varepsilon^{ikj}K^k + \varepsilon^{ijl}K^l) \\ &= i\hbar\varepsilon^{ijk}K^k\ . \end{aligned} \tag{2.51}$$

Similarly,

$$\begin{aligned}[J^1,J^2] &= [M^{23},M^{31}] \\ &= i\hbar(g^{12}M^{33} - g^{33}M^{21} - g^{32}M^{13} + g^{12}M^{33}) \\ &= i\hbar(0 - M^{21} - 0 + 0) \\ &= i\hbar J^3\ , \end{aligned} \tag{2.52}$$

and cyclic permutations of 123.

2.5) From eq. (2.44), we have

$$U(1{+}\delta\omega)^{-1}P^\mu U(1{+}\delta\omega) = P^\mu + \tfrac{i}{2\hbar}\delta\omega_{\rho\sigma}[P^\mu,M^{\rho\sigma}]\ . \tag{2.53}$$

Also,

$$\begin{aligned}(1{+}\delta\omega)^\mu{}_\rho P^\rho &= P^\mu + \delta\omega^\mu{}_\rho P^\rho \\ &= P^\mu + \delta\omega_{\sigma\rho}g^{\sigma\mu}P^\rho \\ &= P^\mu + \tfrac{1}{2}\delta\omega_{\sigma\rho}(g^{\sigma\mu}P^\rho - g^{\rho\mu}P^\sigma) \\ &= P^\mu - \tfrac{1}{2}\delta\omega_{\rho\sigma}(g^{\sigma\mu}P^\rho - g^{\rho\mu}P^\sigma)\ . \end{aligned} \tag{2.54}$$

Equating the right-hand sides of eqs. (2.53) and (2.54), and matching the coefficients of $\delta\omega_{\rho\sigma}$ gives

$$[P^\mu,M^{\rho\sigma}] = i\hbar(g^{\sigma\mu}P^\rho - g^{\rho\mu}P^\sigma)\ , \tag{2.55}$$

which is equivalent to eq. (2.18).

2.6) Starting with eq. (2.55) we have

$$\begin{aligned}
\tfrac{1}{2}\varepsilon^{ikl}[P^0, M^{kl}] &= \tfrac{1}{2}\varepsilon^{ikl}(i\hbar)(g^{l0}P^k - g^{0k}P^l) \\
&= 0 \,, && (2.56) \\
\tfrac{1}{2}\varepsilon^{ikl}[P^j, M^{kl}] &= \tfrac{1}{2}i\hbar\varepsilon^{ikl}(i\hbar)(g^{lj}P^k - g^{kj}P^l) \\
&= -i\hbar\varepsilon^{ijk}P^k \,, && (2.57) \\
[P^0, M^{i0}] &= i\hbar(g^{00}P^i - g^{i0}P^0) \\
&= i\hbar((-1)P^i - 0) \,, && (2.58) \\
[P^j, M^{i0}] &= i\hbar(g^{0j}P^i - g^{ij}P^0) \\
&= i\hbar(0 - \delta^{ij}P^0) \,. && (2.59)
\end{aligned}$$

Using $J^i = \frac{1}{2}\varepsilon^{ikl}M^{kl}$, $K^i = M^{i0}$, and $H = P^0$, and doing a little rearranging, we get eq. (2.19).

2.7) Translations should add: $T(a_1)T(a_2) = T(a_1 + a_2)$. Then taking a_i to be infinitesimal yields $[P^\mu, P^\nu] = 0$. (This is left as a further exercise.)

2.8) a) Using eq. (2.44), the left-hand side of eq. (2.26) becomes

$$U(1{+}\delta\omega)^{-1}\varphi(x)U(1{+}\delta\omega) = \varphi(x) + \tfrac{i}{2\hbar}\delta\omega_{\mu\nu}[\varphi(x), M^{\mu\nu}] \,. \qquad (2.60)$$

Using $\varphi(x{+}\delta x) = \varphi(x) + \delta x_\nu \partial^\nu \varphi(x)$, the right-hand side of eq. (2.26) becomes

$$\begin{aligned}
\varphi((1{-}\delta\omega)x) &= \varphi(x) - \delta\omega_{\nu\mu}x^\mu\partial^\nu\varphi(x) \\
&= \varphi(x) + \tfrac{1}{2}\delta\omega_{\mu\nu}(x^\mu\partial^\nu - x^\nu\partial^\mu)\varphi(x) \,. && (2.61)
\end{aligned}$$

We now get eq. (2.29) by matching the coefficients of $\delta\omega_{\mu\nu}$ in eqs. (2.60) and (2.61).

b) The key point is that $\mathcal{L}^{\mu\nu}$, a differential operator acting on functions of x, commutes with $M^{\rho\sigma}$, an operator in Hilbert space that is inpedendent of x. Therefore, acting on $[\varphi, M^{\rho\sigma}]$ with $\mathcal{L}^{\mu\nu}$, we get

$$\mathcal{L}^{\mu\nu}[\varphi, M^{\rho\sigma}] = [\mathcal{L}^{\mu\nu}\varphi, M^{\rho\sigma}] \,. \qquad (2.62)$$

On the LHS of eq. (2.62), we use $[\varphi, M^{\rho\sigma}] = \mathcal{L}^{\rho\sigma}\varphi$. On the RHS, we use $\mathcal{L}^{\mu\nu}\varphi = [\varphi, M^{\mu\nu}]$. The result is

$$\mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma}\varphi = [[\varphi, M^{\mu\nu}], M^{\rho\sigma}] \,. \qquad (2.63)$$

c) Terms cancel in pairs when all the commutators are expanded out.

d) Exchanging $\mu\nu \leftrightarrow \rho\sigma$ in eq. (2.63), we get

$$\begin{aligned}
\mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu}\varphi &= [[\varphi, M^{\rho\sigma}], M^{\mu\nu}] \\
&= -[[M^{\rho\sigma}, \varphi], M^{\mu\nu}] \,. && (2.64)
\end{aligned}$$

Subtracting eq. (2.64) from eq. (2.63), we have

$$\begin{aligned}
[\mathcal{L}^{\mu\nu}, \mathcal{L}^{\rho\sigma}]\varphi &= [[\varphi, M^{\mu\nu}], M^{\rho\sigma}] + [[M^{\rho\sigma}, \varphi], M^{\mu\nu}] \\
&= -[[M^{\mu\nu}, M^{\rho\sigma}], \varphi] \\
&= [\varphi, [M^{\mu\nu}, M^{\rho\sigma}]] \,, && (2.65)
\end{aligned}$$

where the second equality follows from the Jacobi identity.

e) We begin with

$$\begin{aligned}(x^\mu\partial^\nu)(x^\rho\partial^\sigma) &= x^\mu(g^{\nu\rho}+x^\rho\partial^\nu)\partial^\sigma \\ &= g^{\nu\rho}x^\mu\partial^\sigma + x^\mu x^\rho\partial^\nu\partial^\sigma \ .\end{aligned} \tag{2.66}$$

Exchanging $\mu\nu \leftrightarrow \rho\sigma$ and subtracting, we get

$$[x^\mu\partial^\nu, x^\rho\partial^\sigma] = g^{\nu\rho}x^\mu\partial^\sigma - g^{\sigma\mu}x^\rho\partial^\nu \ . \tag{2.67}$$

Antisymmetrizing on $\mu \leftrightarrow \nu$, we find

$$\begin{aligned}[x^\mu\partial^\nu - x^\nu\partial^\mu, x^\rho\partial^\sigma] = &+ g^{\nu\rho}x^\mu\partial^\sigma - g^{\sigma\mu}x^\rho\partial^\nu \\ &- g^{\mu\rho}x^\nu\partial^\sigma + g^{\sigma\nu}x^\rho\partial^\mu \ .\end{aligned} \tag{2.68}$$

Antisymmetrizing on $\rho \leftrightarrow \sigma$ and regrouping, we find

$$\begin{aligned}[x^\mu\partial^\nu - x^\nu\partial^\mu, x^\rho\partial^\sigma - x^\sigma\partial^\rho] = &+ g^{\nu\rho}(x^\mu\partial^\sigma - x^\sigma\partial^\mu) \\ &- g^{\sigma\mu}(x^\rho\partial^\nu - x^\nu\partial^\rho) \\ &- g^{\mu\rho}(x^\nu\partial^\sigma - x^\sigma\partial^\nu) \\ &+ g^{\sigma\nu}(x^\rho\partial^\mu - x^\mu\partial^\rho) \ .\end{aligned} \tag{2.69}$$

Using $x^\mu\partial^\nu - x^\nu\partial^\mu = (i/\hbar)\mathcal{L}^{\mu\nu}$, we get

$$[\mathcal{L}^{\mu\nu}, \mathcal{L}^{\rho\sigma}] = i\hbar(-g^{\nu\rho}\mathcal{L}^{\mu\sigma} + g^{\sigma\mu}\mathcal{L}^{\rho\nu} + g^{\mu\rho}\mathcal{L}^{\nu\sigma} - g^{\sigma\nu}\mathcal{L}^{\rho\mu}) \ . \tag{2.70}$$

This is equivalent to eq. (2.47) for the M's.

f) This follows immediately from eqs. (2.47), (2.65), and (2.70), and $[\varphi, M^{\rho\sigma}] = \mathcal{L}^{\rho\sigma}\varphi$. See *Weinberg I* for a proof that there is no central charge.

2.9) a) Eq. (2.27) is equivalent to $U(\Lambda)^{-1}\partial^\rho\varphi(x)U(\Lambda) = \partial^\rho\varphi(\Lambda^{-1}x)$, which is just the derivative of $U(\Lambda)^{-1}\varphi(x)U(\Lambda) = \varphi(\Lambda^{-1}x)$. The infinitesimal form of the latter is $[\varphi, M^{\mu\nu}] = \mathcal{L}^{\mu\nu}\varphi$. Acting with ∂^ρ, we get

$$[\partial^\rho\varphi, M^{\mu\nu}] = \partial^\rho\mathcal{L}^{\mu\nu}\varphi \ . \tag{2.71}$$

Next we use

$$\begin{aligned}[\partial^\rho, \mathcal{L}^{\mu\nu}] &= \tfrac{\hbar}{i}[\partial^\rho, x^\mu\partial^\nu - x^\nu\partial^\mu] \\ &= \tfrac{\hbar}{i}([\partial^\rho, x^\mu]\partial^\nu - [\partial^\rho, x^\nu]\partial^\mu) \\ &= \tfrac{\hbar}{i}(g^{\rho\mu}\partial^\nu - g^{\rho\nu}\partial^\mu) \\ &= \tfrac{\hbar}{i}(g^{\rho\mu}\delta^\nu{}_\tau - g^{\rho\nu}\delta^\mu{}_\tau)\partial^\tau \\ &= (S_{\rm V}^{\mu\nu})^\rho{}_\tau\partial^\tau \ .\end{aligned} \tag{2.72}$$

Thus we have

$$[\partial^\rho\varphi, M^{\mu\nu}] = \mathcal{L}^{\mu\nu}\partial^\rho\varphi + (S_{\rm V}^{\mu\nu})^\rho{}_\tau\partial^\tau\varphi \ . \tag{2.73}$$

b) First, note that $(S_{\rm V}^{\mu\nu})^{\rho}{}_{\rho} = 0$. Next, define $(\mathcal{L}^{\mu\nu})^{\rho}{}_{\tau} \equiv \mathcal{L}^{\mu\nu}\delta^{\rho}{}_{\tau}$. Then, suppressing the matrix indices, we have $[\mathcal{L}^{\mu\nu}, S_{\rm V}^{\rho\sigma}] = 0$. Now define $\mathcal{J}^{\mu\nu} \equiv \mathcal{L}^{\mu\nu} + S_{\rm V}^{\mu\nu}$. Then we have

$$[\partial^{\rho}\varphi, M^{\mu\nu}] = (\mathcal{J}^{\mu\nu})^{\rho}{}_{\tau}\partial^{\tau}\varphi \ . \tag{2.74}$$

Now we can repeat the analysis in problem 2.8 to conclude that the $\mathcal{J}$'s have the same commutation relations as the M's. Since $\mathcal{L}$'s and $S_{\rm V}$'s commute, the $S_{\rm V}$'s by themselves must have the same commutation relations. (Of course, this can be verified with a direct but tedious calculation.)

c) Note that eq. (2.33) yields

$$S_{\rm V}^{12} = \hbar \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & +i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{2.75}$$

Since the first and last row and column are all zeroes, we can focus on the middle rows and columns, and write $S_{\rm V}^{12} = \hbar\sigma_2$, where σ_2 is a Pauli matrix, which has eigenvalues ± 1. For any matrix like σ_2 with eigenvalues ± 1, we have $(\sigma_2)^2 = 1$, and so, by Taylor expansion, $\exp(-i\theta\sigma_2) = (\cos\theta) - i(\sin\theta)\sigma_2$. Thus we have

$$\exp(-i\theta S_{\rm V}^{12}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{2.76}$$

d) Note that eq. (2.33) yields

$$S_{\rm V}^{30} = \frac{\hbar}{i} \begin{pmatrix} 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 \end{pmatrix} . \tag{2.77}$$

Since the middle rows and columns are all zeroes, we can foucs on the first and last row and column, and write $S_{\rm V}^{30} = \hbar\sigma_1$, where σ_1 is a Pauli matrix, which has eigenvalues ± 1. For any matrix like σ_1 with eigenvalues ± 1, we have $(\sigma_1)^2 = 1$, and so, by Taylor expansion, $\exp(\eta\sigma_1) = (\cosh\eta) + (\sinh\eta)\sigma_1$. Thus we have

$$\exp(-i\theta S_{\rm V}^{30}) = \begin{pmatrix} \cosh\eta & 0 & 0 & \sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\eta & 0 & 0 & \cosh\eta \end{pmatrix} . \tag{2.78}$$

3 Canonical Quantization of Scalar Fields

3.1) We begin with

$$a(\mathbf{k}) = \int d^3x \; e^{-ikx}\Big[i\Pi(x) + \omega\varphi(x)\Big] \; ,$$
$$a^\dagger(\mathbf{k}) = \int d^3y \; e^{+iky}\Big[-i\Pi(y) + \omega\varphi(y)\Big] \; . \tag{3.39}$$

Since $a(\mathbf{k})$ and $a^\dagger(\mathbf{k})$ are time independent, we can take $y^0 = x^0$. Then we have

$$\begin{aligned}[a(\mathbf{k}), a(\mathbf{k}')] &= \int d^3x \; d^3y \; e^{-ikx-ik'y}\Big([i\Pi(x), \omega\varphi(y)] + [\omega\varphi(x), i\Pi(y)]\Big) \\ &= \int d^3x \; d^3y \; e^{-ikx-ik'y}\Big(-i^2\omega\delta^3(\mathbf{x}{-}\mathbf{y}) + i^2\omega\delta^3(\mathbf{x}{-}\mathbf{y})\Big) \\ &= 0 \; . \end{aligned} \tag{3.40}$$

Then $[a^\dagger(\mathbf{k}), a^\dagger(\mathbf{k}')] = 0$ follows by hermitian conjugation. Also,

$$\begin{aligned}[a(\mathbf{k}), a^\dagger(\mathbf{k}')] &= \int d^3x \; d^3y \; e^{-ikx+ik'y}\Big([i\Pi(x), \omega\varphi(y)] + [\omega\varphi(x), -i\Pi(y)]\Big) \\ &= \int d^3x \; d^3y \; e^{-ikx+ik'y}\Big(-i^2\omega\delta^3(\mathbf{x}{-}\mathbf{y}) - i^2\omega\delta^3(\mathbf{x}{-}\mathbf{y})\Big) \\ &= 2\omega \int d^3x \; d^3y \; e^{-ikx+ik'y} \, \delta^3(\mathbf{x}{-}\mathbf{y}) \\ &= 2\omega \int d^3x \; e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} e^{+i(k^0-k'^0)x^0} \\ &= 2\omega(2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}') \; . \end{aligned} \tag{3.41}$$

3.2) We begin by noting that

$$\begin{aligned}[a^\dagger(\mathbf{k})a(\mathbf{k}), a^\dagger(\mathbf{k}_1)\ldots a^\dagger(\mathbf{k}_n)] &= [a^\dagger(\mathbf{k})a(\mathbf{k}), a^\dagger(\mathbf{k}_1)]a^\dagger(\mathbf{k}_2)\ldots a^\dagger(\mathbf{k}_n) \\ &\quad + \ldots + a^\dagger(\mathbf{k}_1)a^\dagger(\mathbf{k}_2)\ldots[a^\dagger(\mathbf{k})a(\mathbf{k}), a^\dagger(\mathbf{k}_n)] \\ &= (2\pi)^3 2\omega\delta^3(\mathbf{k}-\mathbf{k}_1)a^\dagger(\mathbf{k})a^\dagger(\mathbf{k}_2)\ldots a^\dagger(\mathbf{k}_n) \\ &\quad + \ldots + (2\pi)^3 2\omega\delta^3(\mathbf{k}-\mathbf{k}_n)a^\dagger(\mathbf{k}_1)a^\dagger(\mathbf{k}_2)\ldots a^\dagger(\mathbf{k}) \; . \end{aligned} \tag{3.42}$$

Multiplying by $\omega(\mathbf{k})$ and integrating over $\widetilde{dk} = d^3k/(2\pi)^2 2\omega$, we find

$$[H, a^\dagger(\mathbf{k}_1)\ldots a^\dagger(\mathbf{k}_n)] = (\omega_1 + \ldots + \omega_n)a^\dagger(\mathbf{k}_1)\ldots a^\dagger(\mathbf{k}_n) \; , \tag{3.43}$$

and hence, since $H|0\rangle = 0$,

$$Ha^\dagger(\mathbf{k}_1)\ldots a^\dagger(\mathbf{k}_n)|0\rangle = (\omega_1 + \ldots + \omega_n)a^\dagger(\mathbf{k}_1)\ldots a^\dagger(\mathbf{k}_n)|0\rangle \; . \tag{3.44}$$

3.3) Define the four-dimensional Fourier transform

$$\tilde\varphi(k) \equiv \int d^4x \; e^{-ikx} \, \varphi(x) \tag{3.45}$$

and its inverse

$$\varphi(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{ikx}\, \tilde{\varphi}(k) \; . \tag{3.46}$$

Since $\varphi(x)$ is hermitian, $\tilde{\varphi}(k)$ obeys $\tilde{\varphi}^\dagger(k) = \tilde{\varphi}(-k)$. We now have

$$\begin{aligned} U(\Lambda)^{-1}\tilde{\varphi}(k)U(\Lambda) &= \int d^4x\; e^{-ikx}\, U(\Lambda)^{-1}\varphi(x)U(\Lambda) \\ &= \int d^4x\; e^{-ikx}\, \varphi(\Lambda^{-1}x) \\ &= \int d^4y\; e^{-ik\Lambda y}\, \varphi(y) \\ &= \int d^4y\; e^{-i(\Lambda^{-1}k)y}\, \varphi(y) \\ &= \tilde{\varphi}(\Lambda^{-1}k) \; . \end{aligned} \tag{3.47}$$

The third equality follows from setting $x = \Lambda y$, and recalling that $|\det \Lambda| = 1$. The fourth follows from $k\Lambda y = k^\mu \Lambda_\mu{}^\nu y_\nu = \Lambda_\mu{}^\nu k^\mu y_\nu = (\Lambda^{-1})^\nu{}_\mu k^\mu y_\nu = (\Lambda^{-1}k)^\nu y_\nu$. The fifth follows from eq. (3.45) with $k \to \Lambda^{-1}k$. Next we note that the usual mode expansion is equivalent to

$$\tilde{\varphi}(k) = 2\pi\delta(k^2{+}m^2)\Big[\theta(k^0)a(\mathbf{k}) + \theta(-k^0)a^\dagger(-\mathbf{k})\Big] \; . \tag{3.48}$$

We can see this by plugging eq. (3.48) into eq. (3.46) and carrying out the integral over k^0. For positive k^0, we then have

$$2\pi\delta(k^2{+}m^2)a(\mathbf{k}) = \tilde{\varphi}(k) \; . \tag{3.49}$$

Making an (orthochronous) Lorentz transformation, we have

$$\begin{aligned} 2\pi\delta(k^2{+}m^2)U(\Lambda)^{-1}a(\mathbf{k})U(\Lambda) &= U(\Lambda)^{-1}\tilde{\varphi}(k)U(\Lambda) \\ &= \tilde{\varphi}(\Lambda^{-1}k) \\ &= 2\pi\delta((\Lambda^{-1}k)^2{+}m^2)a(\Lambda^{-1}\mathbf{k}) \\ &= 2\pi\delta(k^2{+}m^2)a(\Lambda^{-1}\mathbf{k}) \; . \end{aligned} \tag{3.50}$$

The third equality follows from eq. (3.49) with $k \to \Lambda^{-1}k$ (note that $(\Lambda^{-1}k)^0$ is positive if k^0 is positive since Λ is orthochronous), and the fourth from $(\Lambda^{-1}k)^2 = k^2$. Matching the coefficients of the delta function on the LHS and final RHS of eq. (3.50) then yields

$$U(\Lambda)^{-1}a(\mathbf{k})U(\Lambda) = a(\Lambda^{-1}\mathbf{k}) \; , \tag{3.51}$$

with k^0 positive. The hermitian conjugate of eq. (3.51) is

$$U(\Lambda)^{-1}a^\dagger(\mathbf{k})U(\Lambda) = a^\dagger(\Lambda^{-1}\mathbf{k}) \; . \tag{3.52}$$

Finally, we have

$$\begin{aligned} U(\Lambda)a^\dagger(\mathbf{k}_1)\ldots a^\dagger(\mathbf{k}_n)|0\rangle &= U(\Lambda)a^\dagger(\mathbf{k}_1)U(\Lambda)^{-1}\ldots U(\Lambda)a^\dagger(\mathbf{k}_n)U(\Lambda)^{-1}U(\Lambda)|0\rangle \\ &= a^\dagger(\Lambda\mathbf{k}_1)\ldots a^\dagger(\Lambda\mathbf{k}_n)|0\rangle \; , \end{aligned} \tag{3.53}$$

where we used eq. (3.52) and $U(\Lambda)|0\rangle = |0\rangle$.

3.4) a) $T(\delta a) = I - i\delta a_\mu P^\mu$ and $T(\delta a)^{-1} = I + i\delta a_\mu P^\mu$, so

$$T(\delta a)^{-1}\varphi(x)T(\delta a) = \varphi(x) - i\delta a_\mu[\varphi(x), P^\mu] \ . \tag{3.54}$$

Also, by Taylor expansion,

$$\varphi(x-a) = \varphi(x) - \delta a_\mu \partial^\mu \varphi(x) \ . \tag{3.55}$$

Matching the coefficients of δa_μ yields

$$[\varphi(x), P^\mu] = \tfrac{1}{i}\partial^\mu \varphi(x) \ . \tag{3.56}$$

b) Setting $\mu = 0$ and recalling that $P^0 = H$ and $\partial^0 = -\partial_0 = \partial/\partial t$, we get $i\dot{\varphi} = [\varphi, H]$.

c) $H = \frac{1}{2}\int d^3y\,[\Pi^2 + (\nabla\varphi)^2 + m^2\varphi^2]$. Since H is time independent, we can take $y^0 = x^0$. Then

$$\begin{aligned}
[\varphi(x), H] &= \tfrac{1}{2}\int d^3y\,[\varphi(x), \Pi^2(y)] \\
&= \tfrac{1}{2}\int d^3y\,\Big([\varphi(x),\Pi(y)]\Pi(y) + \Pi(y)[\varphi(x),\Pi(y)]\Big) \\
&= \tfrac{1}{2}\int d^3y\,\Big(i\delta^3(\mathbf{x}-\mathbf{y})\Pi(y) + \Pi(y)i\delta^3(\mathbf{x}-\mathbf{y})\Big) \\
&= i\int d^3y\,\delta^3(\mathbf{x}-\mathbf{y})\Pi(y) \\
&= i\Pi(x) \ .
\end{aligned} \tag{3.57}$$

Combining with our result from part (b), we find $\Pi = \dot{\varphi}$. Next,

$$\begin{aligned}
[\Pi(x), H] &= \tfrac{1}{2}\int d^3y\,[\Pi(x), \nabla^i\varphi(y)\nabla^i\varphi(y) + m^2\varphi^2(y)] \\
&= \tfrac{1}{2}\int d^3y\,\Big(\nabla^i_y[\Pi(x),\varphi(y)]\nabla^i\varphi(y) + \nabla^i\varphi(y)\nabla^i_y[\Pi(x),\varphi(y)] \\
&\qquad\qquad + m^2[\Pi(x),\varphi(y)]\varphi(y) + m^2\varphi(y)[\Pi(x),\varphi(y)]\Big) \\
&= -i\int d^3y\,\Big(\nabla^i_y\delta^3(\mathbf{x}-\mathbf{y})\nabla^i\varphi(y) + m^2\delta^3(\mathbf{x}-\mathbf{y})\varphi(y)\Big) \\
&= -i\int d^3y\,\Big(-\delta^3(\mathbf{x}-\mathbf{y})\nabla^2\varphi(y) + m^2\delta^3(\mathbf{x}-\mathbf{y})\varphi(y)\Big) \\
&= -i(-\nabla^2 + m^2)\varphi(x) \ .
\end{aligned} \tag{3.58}$$

The Heisenberg equation for Π, $[\Pi, H] = i\dot{\Pi}$, then yields $\dot{\Pi} = -(-\nabla^2 + m^2)\varphi$. Since $\Pi = \dot{\varphi}$, this is equivalent to $\ddot{\varphi} = -(-\nabla^2 + m^2)\varphi$, which is the Klein-Gordon equation.

d) We have

$$\begin{aligned}
[\varphi(x), \mathbf{P}] &= -\int d^3y\,[\varphi(x),\Pi(y)]\nabla\varphi(y) \\
&= -i\int d^3y\,\delta^3(\mathbf{x}-\mathbf{y})\nabla\varphi(y) \\
&= -i\nabla\varphi(x) \ ,
\end{aligned} \tag{3.59}$$

which agrees with eq. (3.56).

e) We have

$$\begin{aligned}\mathbf{P} &= -\int d^3x\,\Pi(x)\nabla\varphi(x)\\ &= -\int \widetilde{dk}\,\widetilde{dk}'\,d^3x\,\Big(-i\omega\,a(\mathbf{k})e^{ikx} + i\omega\,a^\dagger(\mathbf{k})e^{-ikx}\Big)\Big(+i\mathbf{k}'\,a(\mathbf{k}')e^{ik'x} - i\mathbf{k}'\,a^\dagger(\mathbf{k}')e^{-ik'x}\Big)\\ &= -(2\pi)^3\int \widetilde{dk}\,\widetilde{dk}'\,\Big[\delta^3(\mathbf{k}-\mathbf{k}')(-\omega\mathbf{k}')\Big(a^\dagger(\mathbf{k})a(\mathbf{k}')e^{-i(\omega-\omega')t} + a(\mathbf{k})a^\dagger(\mathbf{k}')e^{+i(\omega-\omega')t}\Big)\\ &\qquad\qquad + \delta^3(\mathbf{k}+\mathbf{k}')(+\omega\mathbf{k}')\Big(a(\mathbf{k})a(\mathbf{k}')e^{-i(\omega+\omega')t} + a^*(\mathbf{k})a^*(\mathbf{k}')e^{+i(\omega+\omega')t}\Big)\\ &= \tfrac{1}{2}\int \widetilde{dk}\,\mathbf{k}\,\Big[a^\dagger(\mathbf{k})a(\mathbf{k}) + a(\mathbf{k})a^\dagger(\mathbf{k}) + a(\mathbf{k})a(-\mathbf{k})e^{-2i\omega t} + a^\dagger(\mathbf{k})a^\dagger(-\mathbf{k})e^{+2i\omega t}\Big].\end{aligned} \tag{3.60}$$

Note that the third term, $\mathbf{k}\,a(\mathbf{k})a(-\mathbf{k})e^{-2i\omega t}$, is odd under $\mathbf{k}\leftrightarrow-\mathbf{k}$, and hence vanishes when integrated over $\widetilde{dk}$; the same is true of its hermitian conjugate (the fourth term). Also, in the second term, we can write $aa^\dagger = a^\dagger a +$ constant; the constant term, after being multiplied by $\mathbf{k}$, also vanishes when integrated over $\widetilde{dk}$. We therefore get

$$\mathbf{P} = \int \widetilde{dk}\,\mathbf{k}\,a^\dagger(\mathbf{k})a(\mathbf{k})\ , \tag{3.61}$$

which is just what we expect.

3.5) a) $\delta S = \int d^4x\,(+\partial^2\varphi - m^2\varphi)\delta\varphi^\dagger + \text{h.c.}$ after integrating by parts; the coefficients of both $\delta\varphi$ and $\delta\varphi^\dagger$ must vanish.

b) $\Pi \equiv \partial\mathcal{L}/\partial\dot\varphi = \dot\varphi^\dagger$, $\Pi^\dagger \equiv \partial\mathcal{L}/\partial\dot\varphi^\dagger = \dot\varphi$, $\mathcal{H} = \Pi\dot\varphi + \Pi^\dagger\dot\varphi^\dagger - \mathcal{L} = \Pi^\dagger\Pi + \nabla\varphi^\dagger\nabla\varphi + m^2\varphi^\dagger\varphi$.

c) Following the text, we get $a(\mathbf{k}) = i\int d^3x\ e^{-ikx}\,\overleftrightarrow{\partial_0}\,\varphi(x)$. Then we note that exchanging $\varphi\leftrightarrow\varphi^\dagger$ is equivalent to $a(\mathbf{k})\leftrightarrow b(\mathbf{k})$, and so $b(\mathbf{k}) = i\int d^3x\ e^{-ikx}\,\overleftrightarrow{\partial_0}\,\varphi^\dagger(x)$.

d) This is straightforward but tedious; the answer is the expected one: $[a,a]=[b,b]=[a,b^\dagger]=[a^\dagger,b]=0$, $[a,a^\dagger]=[b,b^\dagger]\sim\delta$.

e) Again, straightforward but tedious, just like the derivation in the text; the final result is

$$H = \int \widetilde{dk}\,\omega\,\Big[a^\dagger(\mathbf{k})a(\mathbf{k}) + b^\dagger(\mathbf{k})b(\mathbf{k})\Big] + (2\mathcal{E}_0 - \Omega_0)V\ , \tag{3.62}$$

with $\mathcal{E}_0 = \frac{1}{2}(2\pi)^{-3}\int d^3k\ \omega$; each set of oscillators (a and b) contributes $\mathcal{E}_0 V$ to the zero-point energy.

4 The Spin-Statistics Theorem

4.1) We want to evaluate the Lorentz-invariant integral $\int \widetilde{dk}\, e^{ik(x-x')}$, where $(x-x')^2 \equiv r^2 > 0$. There is then a frame where $t' = t$, and we work in that frame. Then we have

$$\begin{aligned}
\int \widetilde{dk}\, e^{ik(x-x')} &= \int \frac{d^3k}{(2\pi)^3 2\omega}\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \\
&= \frac{2\pi}{2(2\pi)^3} \int_0^\infty \frac{dk\, k^2}{\omega} \int_{-1}^{+1} d\cos\theta\; e^{ikr\cos\theta} \\
&= \frac{1}{8\pi^2} \int_0^\infty \frac{dk\, k^2}{\omega}\, \frac{2\sin(kr)}{kr} \\
&= \frac{1}{4\pi^2 r} \int_0^\infty dk\, \frac{k\sin(kr)}{(k^2+m^2)^{1/2}} \\
&= \frac{1}{4\pi^2 r}\, mK_1(mr)\;, \qquad (4.16)
\end{aligned}$$

where $K_1(z)$ is a modified Bessel function. As $z \to 0$, $zK_1(z) \to 1$, and so as $m \to 0$, the right-hand side of eq. (4.16) becomes $1/4\pi^2 r^2$.

5 The LSZ Reduction Formula

5.1) From our results in solution 3.5c, and taking hermitian conjugates as well, we have

$$\begin{aligned}
a(\mathbf{k}) &= i\int d^3x\; e^{-ikx}\,\overleftrightarrow{\partial_0}\,\varphi(x)\;, \\
b(\mathbf{k}) &= i\int d^3x\; e^{-ikx}\,\overleftrightarrow{\partial_0}\,\varphi^\dagger(x)\;, \\
a^\dagger(\mathbf{k}) &= -i\int d^3x\; e^{+ikx}\,\overleftrightarrow{\partial_0}\,\varphi^\dagger(x)\;, \\
b^\dagger(\mathbf{k}) &= -i\int d^3x\; e^{+ikx}\,\overleftrightarrow{\partial_0}\,\varphi(x)\;,
\end{aligned} \tag{5.28}$$

Following the analysis in the text, this yields the following replacements inside the time-ordered product:

$$\begin{aligned}
a_{1'}(+\infty) &\to i\int d^4x'_1\; e^{-ik'_1x'_1}(-\partial^2_{1'}+m^2)\varphi(x'_1)\;, \\
b_{2'}(+\infty) &\to i\int d^4x'_2\; e^{-ik'_2x'_2}(-\partial^2_{2'}+m^2)\varphi^\dagger(x'_2)\;, \\
a^\dagger_1(-\infty) &\to i\int d^4x_1\; e^{+ik_1x_1}(-\partial^2_{1}+m^2)\varphi^\dagger(x_1)\;, \\
b^\dagger_2(-\infty) &\to i\int d^4x_2\; e^{+ik_2x_2}(-\partial^2_{2}+m^2)\varphi(x_2)\;.
\end{aligned} \tag{5.29}$$

We see that outgoing a particles and incoming b particles result in a φ, and that outgoing b particles and incoming a particles result in a $\varphi^\dagger$. Outgoing particles get a phase factor with a plus sign in the exponent, and incoming particles get a phase factor with a minus sign in the exponent.

6 Path Integrals in Quantum Mechanics

6.1) a) We wish to perform the integral

$$I_N \equiv \prod_{j=0}^{N} \int \frac{dp_j}{2\pi}\, e^{ip_j a_j}\, e^{-cp_j^2/2}\,, \tag{6.24}$$

where $a_j \equiv q_{j+1} - q_j \equiv \dot{q}_j \delta t$ and $c \equiv i\delta t/m$. The result is

$$\begin{aligned} I_N &= \prod_{j=0}^{N} \frac{e^{-a_j^2/2c}}{(2\pi c)^{1/2}} \\ &= \left(\frac{m}{2\pi i \delta t}\right)^{(N+1)/2} \exp\!\left(i\delta t \sum_j \tfrac{1}{2} m \dot{q}_j^2\right). \end{aligned} \tag{6.25}$$

Therefore $\mathcal{D}q = C \prod_{j=1}^{N} dq_j$ with $C = (m/2\pi i \delta t)^{(N+1)/2}$.

b) We now have

$$\begin{aligned} \langle q'', t'' | q', t' \rangle &= \left(\frac{m}{2\pi i \delta t}\right)^{(N+1)/2} \prod_{k=1}^{N} dq_k \exp\!\left(\frac{im}{2\delta t} \sum_{j=0}^{N} (q_{j+1} - q_j)^2\right) \\ &= \left(\frac{1}{2\pi c}\right)^{(N+1)/2} \prod_{k=1}^{N} dq_k \exp\!\left(-\sum_{j=0}^{N} \frac{(q_{j+1} - q_j)^2}{2c}\right). \end{aligned} \tag{6.26}$$

The integral over q_1 is

$$\int dq_1\, e^{-(q_2-q_1)^2/2c} e^{-(q_1-q_0)^2/2c} = \left(\tfrac{1}{2}(2\pi c)\right)^{1/2} e^{-(q_2-q_0)^2/4c}\,. \tag{6.27}$$

The integral over q_2 is now

$$\int dq_2\, e^{-(q_3-q_2)^2/2c} e^{-(q_2-q_0)^2/4c} = \left(\tfrac{2}{3}(2\pi c)\right)^{1/2} e^{-(q_3-q_0)^2/6c}\,. \tag{6.28}$$

In general, the integral over q_N is

$$\int dq_N\, e^{-(q_{N+1}-q_N)^2/2c} e^{-(q_N-q_0)^2/2Nc} = \left(\tfrac{N}{N+1}(2\pi c)\right)^{1/2} e^{-(q_{N+1}-q_0)^2/2(N+1)c}\,. \tag{6.29}$$

Therefore we have

$$\prod_{k=1}^{N} dq_k \exp\!\left(-\sum_{j=0}^{N} \frac{(q_{j+1} - q_j)^2}{2c}\right) = \left(\tfrac{1}{N+1}\right)^{1/2} \left(2\pi c\right)^{N/2} e^{-(q_{N+1}-q_0)^2/2(N+1)c}\,, \tag{6.30}$$

and so

$$\begin{aligned} \langle q'', t'' | q', t' \rangle &= \left(\frac{1}{2\pi c(N+1)}\right)^{1/2} e^{-(q_{N+1}-q_0)^2/2(N+1)c} \\ &= \left(\frac{m}{2\pi i(t''-t')}\right)^{1/2} e^{im(q''-q')^2/2(t''-t')}\,, \end{aligned} \tag{6.31}$$

where we used $c = i\delta t/m$ and $(N{+}1)\delta t = t''{-}t'$ to get the second line. The exponent must be dimensionless, and since $\langle q''|q'\rangle = \delta(q''-q')$, the prefactor must have dimensions of inverse length; therefore

$$\langle q'',t''|q',t'\rangle = \left(\frac{m}{2\pi i\hbar(t''{-}t')}\right)^{1/2} e^{im(q''-q')^2/2\hbar(t''-t')} \ . \tag{6.32}$$

c) Let $T = t''-t'$; we have $H = \frac{1}{2m}P^2$, and so

$$\begin{aligned}
\langle q'',t''|q',t'\rangle &= \langle q''|e^{-iHT}|q'\rangle \\
&= \int_{-\infty}^{+\infty} dp\, \langle q''|e^{-iHT}|p\rangle\langle p|q'\rangle \\
&= \int_{-\infty}^{+\infty} dp\, \langle q''|p\rangle\langle p|q'\rangle\, e^{-i(p^2/2m)T} \\
&= \int_{-\infty}^{+\infty} dp\, \frac{e^{ipq''}}{\sqrt{2\pi}}\,\frac{e^{-ipq'}}{\sqrt{2\pi}}\, e^{-i(p^2/2m)T} \\
&= \int_{-\infty}^{+\infty} \frac{dp}{2\pi}\, e^{ip(q''-q')-i(T/m)p^2/2} \\
&= \left(\frac{m}{2\pi iT}\right)^{1/2} e^{im(q''-q')^2/2T} \ ,
\end{aligned} \tag{6.33}$$

which agrees (as it should!) with eq. (6.31).

7 The Path Integral for the Harmonic Oscillator

7.1) Setting $t' = 0$ for notational convenience, we have

$$\begin{aligned} G(t) &= \int_{-\infty}^{+\infty} \frac{dE}{2\pi} \, \frac{e^{-iEt}}{-E^2 + \omega^2 - i\epsilon} \\ &= \int_{-\infty}^{+\infty} dE \, \frac{-e^{-iEt}/2\pi}{(E-(\omega - i\epsilon))(E+(w - i\epsilon))} \end{aligned} \tag{7.19}$$

Think of E as a complex variable. If $t > 0$, we can add to eq. (7.19) an integral along an arc at infinity in the lower half complex E-plane, since e^{-iEt} vanishes on this arc. This produces a closed contour that encircles the pole at $E = \omega - i\epsilon$ in a clockwise direction. The residue of this pole is $(-e^{-i(\omega - i\epsilon)t}/2\pi)/(2(\omega - i\epsilon)) \to -e^{-i\omega t}/4\pi\omega$ as $\epsilon \to 0$. By the residue theorem, the value of the integral is $-2\pi i$ times this residue. Similarly, if $t < 0$, we can add an arc at infinity in the upper-half plane. This produces a closed contour that encircles the pole at $E = -(\omega - i\epsilon)$ in a counterclockwise direction. The residue of this pole is $(-e^{i(\omega - i\epsilon)t}/2\pi)/(-2(\omega - i\epsilon)) \to e^{i\omega t}/4\pi\omega$ as $\epsilon \to 0$. By the residue theorem, the value of the integral is $+2\pi i$ times this residue. Combining these two cases, we have

$$G(t) = \tfrac{i}{2\omega} e^{-i\omega|t|} \; . \tag{7.20}$$

7.2) We wish to show that $G(t)$, as given by eq. (7.20), obeys $\ddot{G} + \omega^2 G = \delta(t)$. We first note that $(d/dt)|t| = \operatorname{sign} t$ and $(d/dt) \operatorname{sign} t = 2\delta(t)$. We then have

$$\dot{G}(t) = \tfrac{1}{2} e^{-i\omega|t|} \operatorname{sign}(t) \; , \tag{7.21}$$

and

$$\begin{aligned} \ddot{G}(t) &= -\tfrac{1}{2} i\omega e^{-i\omega|t|} \operatorname{sign}^2(t) + e^{-i\omega|t|}\delta(t) \\ &= -\tfrac{1}{2} i\omega e^{-i\omega|t|} + \delta(t) \\ &= -\omega^2 G(t) + \delta(t) \; . \end{aligned} \tag{7.22}$$

7.3) a) $\dot{Q} = i[H, Q] = i[\frac{1}{2}P^2, Q] = P$ and $\dot{P} = i[H, P] = i[\frac{1}{2}\omega^2 Q^2, P] = -\omega^2 Q$. The solution is

$$\begin{aligned} Q(t) &= Q \cos\omega t + \tfrac{1}{\omega} P \sin\omega t \\ P(t) &= P \cos\omega t - \omega Q \sin\omega t \; . \end{aligned} \tag{7.23}$$

b) $Q = \frac{1}{\sqrt{2\omega}}(a^\dagger + a)$ and $P = i\sqrt{\frac{\omega}{2}}(a^\dagger - a)$; this is a standard result in quantum mechanics (with $\hbar = m = 1$). Plugging these into eq. (7.23) and simplifying, we find

$$\begin{aligned} Q(t) &= \tfrac{1}{\sqrt{2\omega}} \Big(a^\dagger e^{+i\omega t} + a e^{-i\omega t} \Big) \; , \\ P(t) &= i\sqrt{\tfrac{\omega}{2}} \Big(a^\dagger e^{+i\omega t} - a e^{-i\omega t} \Big) \; . \end{aligned} \tag{7.24}$$

c) Assume $t_1 > t_2$. Then we have

$$\begin{aligned}\langle 0|\mathrm{T}Q(t_1)Q(t_2)|0\rangle &= \langle 0|Q(t_1)Q(t_2)|0\rangle \\ &= \tfrac{1}{2\omega}\langle 0|(a^\dagger e^{+i\omega t_1} + ae^{-i\omega t_1})(a^\dagger e^{+i\omega t_2} + ae^{-i\omega t_2})|0\rangle \\ &= \tfrac{1}{2\omega}e^{-i\omega t_1}e^{+i\omega t_2}\langle 0|aa^\dagger|0\rangle \\ &= \tfrac{1}{2\omega}e^{-i\omega(t_1-t_2)}\ . \end{aligned} \tag{7.25}$$

Of course, for $t_1 < t_2$, we have $\langle 0|\mathrm{T}Q(t_1)Q(t_2)|0\rangle = \langle 0|Q(t_2)Q(t_1)|0\rangle = \frac{1}{2\omega}e^{-i\omega(t_2-t_1)}$. Comparing with eq. (7.20), we see that $\langle 0|\mathrm{T}Q(t_1)Q(t_2)|0\rangle = \frac{1}{i}G(t_1-t_2)$.

We can similarly analyze the case of four Q's; for $t_1 > t_2 > t_3 > t_4$, we have

$$\begin{aligned}&\langle 0|\mathrm{T}Q(t_1)Q(t_2)Q(t_3)Q(t_4)|0\rangle \\ &\quad= \tfrac{1}{(2\omega)^2}e^{-i\omega(t_1-t_4)}\langle 0|a(a^\dagger e^{+i\omega t_2} + ae^{-i\omega t_2})(a^\dagger e^{+i\omega t_3} + ae^{-i\omega t_3})a^\dagger|0\rangle \\ &\quad= \tfrac{1}{(2\omega)^2}e^{-i\omega(t_1-t_4)}\big[e^{-i\omega(t_3-t_2)}\langle 0|aa^\dagger aa^\dagger|0\rangle + e^{-i\omega(t_2-t_3)}\langle 0|aaa^\dagger a^\dagger|0\rangle\big] \\ &\quad= \tfrac{1}{(2\omega)^2}e^{-i\omega(t_1-t_4)}\big[e^{-i\omega(t_3-t_2)} + 2e^{-i\omega(t_2-t_3)}\big] \\ &\quad= \tfrac{1}{(2\omega)^2}\big[e^{-i\omega(t_1-t_2)}e^{-i\omega(t_3-t_4)} + e^{-i\omega(t_1-t_3)}e^{-i\omega(t_2-t_4)} + e^{-i\omega(t_1-t_4)}e^{-i\omega(t_2-t_3)}\big] \\ &\quad= \tfrac{1}{i^2}[G(t_1{-}t_2)G(t_3{-}t_4) + G(t_1{-}t_3)G(t_2{-}t_4) + G(t_1{-}t_4)G(t_2{-}t_3)]\ . \end{aligned} \tag{7.26}$$

Other time-orderings follow by relabeling.

7.4) Eq. (7.10) reads

$$\langle 0|0\rangle_f = \exp\left[\frac{i}{2}\int_{-\infty}^{+\infty}\frac{dE}{2\pi}\,\frac{\widetilde{f}(E)\widetilde{f}(-E)}{-E^2+\omega^2-i\epsilon}\right]. \tag{7.27}$$

In general, if $A = \exp(iB)$, then $|A|^2 = \exp(-2\,\mathrm{Im}\,B)$. Since $f(t)$ is real, its Fourier transform $\widetilde{f}(E)$ obeys $\widetilde{f}^*(E) = \widetilde{f}(-E)$; therefore $\widetilde{f}(E)\widetilde{f}(-E) = |\widetilde{f}(E)|^2$, which is purely real. We then use $\mathrm{Im}\,1/(x{-}i\epsilon) = \epsilon/(x^2{+}\epsilon^2)$; as $\epsilon \to 0$, this becomes $\pi\delta(x)$. Thus we have

$$|\langle 0|0\rangle_f|^2 = \exp\left[-\frac{1}{2}\int_{-\infty}^{+\infty}dE\,|\widetilde{f}(E)|^2\,\delta(-E^2{+}\omega^2)\right]. \tag{7.28}$$

Using $\delta(-E^2{+}\omega^2) = \delta(E^2{-}\omega^2) = \frac{1}{2\omega}[\delta(E{-}\omega) + \delta(E{+}\omega)]$, and $|\widetilde{f}(E)|^2 = |\widetilde{f}(-E)|^2$, we get

$$|\langle 0|0\rangle_f|^2 = \exp\Big[-\tfrac{1}{2\omega}|\widetilde{f}(\omega)|^2\Big]\ . \tag{7.29}$$

8 The Path Integral for Free Field Theory

8.1) $(-\partial_x^2+m^2)e^{ikx}=(k^2+m^2)e^{ikx}$, and $(k^2+m^2)/(k^2+m^2-i\epsilon)\to 1$ as $\epsilon\to 0$. The remaining integral yields $\delta^4(x-x')$.

8.2) The first line of eq. (8.13) follows immediately from the solution to problem 7.1. To get the second term on the second line of eq. (8.13) into the form shown, we replace the integration variable $\mathbf{k}$ with $-\mathbf{k}$.

8.3) Set $x'=0$ for notational convenience. Then $(-\nabla^2+m^2)e^{\pm ikx}=(\mathbf{k}^2+m^2)e^{\pm ikx}=\omega^2 e^{\pm ikx}$. Also, $\partial_0[i\theta(t)e^{-i\omega t}]=[i\delta(t)+\omega\theta(t)]e^{-i\omega t}=i\delta(t)+\omega\theta(t)e^{-i\omega t}$, and so $(\partial_0^2+\omega^2)[i\theta(t)e^{-i\omega t}]=i\dot\delta(t)+\omega\delta(t)e^{-i\omega t}=i\dot\delta(t)+\omega\delta(t)$. Similarly, $(\partial_0^2+\omega^2)[i\theta(-t)e^{+i\omega t}]=-i\dot\delta(t)+\omega\delta(t)$. Doing the integral over $\mathbf{k}$, we find

$$\begin{aligned}(-\partial^2+m^2)\Big[i\theta(t)\int\widetilde{dk}\;e^{ikx}\Big] &= +i\dot\delta(t)C(r)+\tfrac12\delta(t)\delta^3(\mathbf{x})\;,\\ (-\partial^2+m^2)\Big[i\theta(-t)\int\widetilde{dk}\;e^{-ikx}\Big] &= -i\dot\delta(t)C(r)+\tfrac12\delta(t)\delta^3(\mathbf{x})\;,\end{aligned}\tag{8.20}$$

where $C(r)$ is defined in eq. (4.12). Adding, we get

$$(-\partial^2+m^2)\Big[i\theta(t)\int\widetilde{dk}\;e^{ikx}+i\theta(-t)\int\widetilde{dk}\;e^{-ikx}\Big]=\delta(t)\delta^3(\mathbf{x})\;.\tag{8.21}$$

8.4) For $t_1>t_2$, we have

$$\begin{aligned}\langle 0|\mathrm{T}\varphi(x_1)\varphi(x_2)|0\rangle &= \int\widetilde{dk}_1\,\widetilde{dk}_2\;\langle 0|\Big(a(\mathbf{k}_1)e^{ik_1x_1}+a^\dagger(\mathbf{k}_1)e^{-ik_1x_1}\Big)\\ &\qquad\times\Big(a(\mathbf{k}_2)e^{ik_2x_2}+a^\dagger(\mathbf{k}_2)e^{-ik_2x_2}\Big)|0\rangle\\ &= \int\widetilde{dk}_1\,\widetilde{dk}_2\;e^{i(k_1x_1-k_2x_2)}\,\langle 0|a(\mathbf{k}_1)a^\dagger(\mathbf{k}_2)|0\rangle\\ &= \int\widetilde{dk}_1\,\widetilde{dk}_2\;e^{i(k_1x_1-k_2x_2)}\,(2\pi)^3\,2\omega_2\,\delta^3(\mathbf{k}_1-\mathbf{k}_2)\\ &= \int\widetilde{dk}_1\,d^3k_2\;e^{i(k_1x_1-k_2x_2)}\,\delta^3(\mathbf{k}_1-\mathbf{k}_2)\\ &= \int\widetilde{dk}_1\;e^{ik_1(x_1-x_2)}\;.\end{aligned}\tag{8.22}$$

Obviously, if $t_2>t_1$, we swap 1 and 2. This yields the last line of eq. (8.15).

8.5) For $x^0>y^0$, we must close the contour in the lower-half k^0 plane. The result will vanish if both poles are above the real k^0 axis, so this is the prescription that yields $\Delta_{\rm ret}(x-y)$. We can implement this prescription via

$$\begin{aligned}\Delta_{\rm ret}(x-y) &= \int\frac{d^4k}{(2\pi)^4}\,\frac{e^{ik(x-y)}}{-(k^0-i\epsilon)^2+\mathbf{k}^2+m^2}\\ &= \int\frac{d^4k}{(2\pi)^4}\,\frac{e^{ik(x-y)}}{k^2+m^2+2ik^0\epsilon}\\ &= \int\frac{d^4k}{(2\pi)^4}\,\frac{e^{ik(x-y)}}{k^2+m^2+i\,\mathrm{sign}(k^0)\epsilon}\;,\end{aligned}\tag{8.23}$$

where the last line follows because only the sign of the infinitesimal term matters (and not its magnitude). Similarly,

$$\Delta_{\rm adv}(x-y) = \int \frac{d^4k}{(2\pi)^4}\,\frac{e^{ik(x-y)}}{k^2+m^2-i\,{\rm sign}(k^0)\epsilon}\;. \tag{8.24}$$

8.6) See the solution to problem 7.4 for more details. We use

$$\begin{aligned}\frac{1}{x-i\epsilon} &= \frac{x}{x^2+\epsilon^2}+\frac{i\epsilon}{x^2+\epsilon^2} \\ &= P\,\frac{1}{x}+i\pi\delta(x)\;,\end{aligned} \tag{8.25}$$

where P denotes the *principal part*. We note that $\widetilde J(k)\widetilde J(-k)=|\widetilde J(k)|^2$ is real, and so

$$\begin{aligned}{\rm Re}\,W_0(J) &= \frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\,|\widetilde J(k)|^2\,P\,\frac{1}{k^2+m^2}\;, \\ {\rm Im}\,W_0(J) &= \frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\,|\widetilde J(k)|^2\,\pi\delta(k^2+m^2) \\ &= \tfrac{1}{2}\int\widetilde{dk}\;|\widetilde J(k)|^2\;.\end{aligned} \tag{8.26}$$

8.7) This is a straightforward generalization; the final result is

$$Z_0(J^\dagger,J) = \exp\!\left[i\!\int\! d^4x\,d^4x'\;J^\dagger(x)\Delta(x-x')J(x')\right]. \tag{8.27}$$

The generalization of eq. (8.14) is

$$\langle 0|{\rm T}\varphi(x_1)\ldots\varphi^\dagger(y_1)\ldots|0\rangle = \frac{1}{i}\,\frac{\delta}{\delta J^\dagger(x_1)}\ldots\frac{1}{i}\,\frac{\delta}{\delta J(y_1)}\ldots Z_0(J^\dagger,J)\Big|_{J^\dagger=J=0}\;. \tag{8.28}$$

This yields

$$\begin{aligned}\langle 0|{\rm T}\varphi(x_1)\varphi(x_2)|0\rangle &= 0\;, \\ \langle 0|{\rm T}\varphi^\dagger(y_1)\varphi^\dagger(y_2)|0\rangle &= 0\;, \\ \langle 0|{\rm T}\varphi(x_1)\varphi^\dagger(y_2)|0\rangle &= \tfrac{1}{i}\Delta(x_1-y_2)\;.\end{aligned} \tag{8.29}$$

From the mode expansions, $\varphi\sim a+b^\dagger$ and $\varphi^\dagger\sim a^\dagger+b$, and the fact that we can get a nonzero result for $\langle 0|\ldots|0\rangle$ only if $\ldots$ contains $aa^\dagger$ or $bb^\dagger$, we see that $\langle 0|\varphi\varphi|0\rangle$ and $\langle 0|\varphi^\dagger\varphi^\dagger|0\rangle$ must vanish, and that $\langle 0|\varphi\varphi^\dagger|0\rangle$ is the same as in the case of a real field. The generalization of eq. (8.17) is then

$$\langle 0|{\rm T}\varphi(x_1)\ldots\varphi(x_n)\varphi^\dagger(y_1)\ldots\varphi^\dagger(y_n)|0\rangle = \frac{1}{i^n}\sum_{\rm perms}\Delta(x_1{-}y_{i_1})\ldots\Delta(x_n{-}y_{i_n})\;, \tag{8.30}$$

where the sum is over permutations of the y_i's. The result vanishes if the number of φ's does not equal the number of $\varphi^\dagger$'s.

8.8) This is a straightforward generalization of the result for a harmonic oscillator. If we put the system in a box (with, say, periodic boundary conditions), then the momentum is discrete, and we can write $\varphi(x) = \sum_{\mathbf{k}} \widetilde{\varphi}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}$, and similarly for $\Pi(\mathbf{x})$; we take $t = 0$. Then

$$H = \tfrac{1}{2} \sum_{\mathbf{k}} \Big[\widetilde{\Pi}_{\mathbf{k}} \widetilde{\Pi}_{-\mathbf{k}} + (\mathbf{k}^2 + m^2) \widetilde{\varphi}_{\mathbf{k}} \widetilde{\varphi}_{-\mathbf{k}} \Big] \,, \tag{8.31}$$

which is just a sum of individual oscillators labeled by $\mathbf{k}$, with $\omega_{\mathbf{k}} = (\mathbf{k}^2 + m^2)^{1/2}$. Thus the ground-state wave function is just a product of the individual wave functions,

$$\psi \propto \prod_{\mathbf{k}} \exp\Big(-\tfrac{1}{2} \omega_{\mathbf{k}} \tilde{A}_{\mathbf{k}} \tilde{A}_{-\mathbf{k}} \Big) \,. \tag{8.32}$$

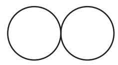
We can replace the product with a sum in the exponent; in the limit of infinite box size, $\sum_{\mathbf{k}} \to \int d^3k/(2\pi)^3$, which yields eq (8.19).

9 The Path Integral for Interacting Field Theory

9.1) See the figures in the text.

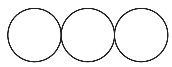
9.2) a) The vertex joins four line segments. The vertex factor is $(4!)(i)(-\lambda/24) = -i\lambda$.

b) $E = 0$, $V = 1$:

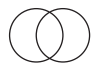


$S = 2^3$

$E = 0$, $V = 2$:



$S = 2^4$

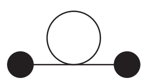


$S = 2 \times 4!$

$E = 2$, $V = 0$:

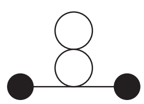
$S = 2$

$E = 2$, $V = 1$:

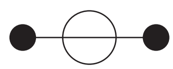


$S = 2^2$

$E = 2$, $V = 2$:



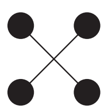
$S = 2^3$



$S = 2 \times 3!$

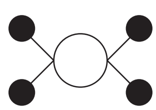
$S = 2^3$

$E = 4$, $V = 1$:

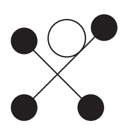


$S = 4!$

$E = 4$, $V = 2$:



$S = 2^4$



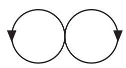
$S = 2 \times 3!$

There are no diagrams with E odd, since it is impossible to draw such a diagram when the vertices connect an even number of lines.

c) Since there is no diagram with a single source, the VEV of φ vanishes. (For another explanation, see section 23.)

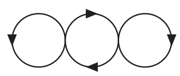
9.3) a) The vertex joins four line segments, two with arrows pointing towards the vertex, and two with arrows pointing away from the vertex; this is because the interaction term involves two φ's and two $\varphi^\dagger$'s. The vertex factor is $(2!)(2!)(i)(-\lambda/4) = -i\lambda$.

b) $E = 0$, $V = 1$:

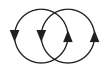


$S = 2$

$E = 0$, $V = 2$:



$S = 2$

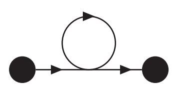


$S = 2^3$

$E = 2$, $V = 0$:

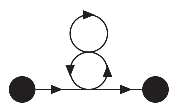
$S = 1$

$E = 2$, $V = 1$:

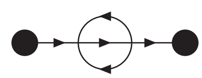


$S = 1$

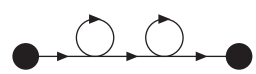
$E = 2$, $V = 2$:



$S = 1$

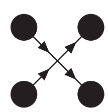


$S = 2$



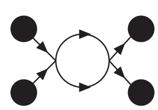
$S = 1$

$E = 4$, $V = 1$:

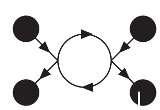


$S = 2^2$

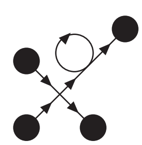
$E = 4$, $V = 2$:



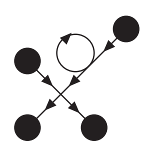
$S = 2^3$



$S = 2$



$S = 2$



$S = 2$

9.4) a) $\exp W(g, J)$ is the path integral for φ^3 theory in $d = 0$ spacetime dimensions, but without the prefactor of i in the exponent. (This means we are working in $d = 0$ *euclidean* spacetime dimensions; see section 29.) In the Feynman-diagram expansion, each propagator is $(2\pi)^{-1/2}\int_{-\infty}^{+\infty} dx\, e^{-x^2/2}x^2 = 1$, each vertex is g, and each source is J. Only the symmetry factor of each diagram is nontrivial. The sum rule $C_{V,E} = \sum_I \frac{1}{S_I}$ follows immediately.

b) We expand $e^{gx^3/6+Jx}$ in powers of g out to g^4, and in powers of J out to J^5; the J^5 term is needed for part (d). Then, odd powers of x integrate to zero, and even powers x^{2n} to $(2n-1)!!$. The result is

$$\begin{aligned} e^{W(g,J)} = {} & 1 + (\tfrac{5}{24}g^2 + \tfrac{385}{1152}g^4) + (\tfrac{1}{2}g + \tfrac{35}{48}g^3)J + (\tfrac{1}{2} + \tfrac{35}{48}g^2 + \tfrac{5005}{2304}g^4)J^2 \\ & + (\tfrac{5}{12}g + \tfrac{385}{288}g^3)J^3 + (\tfrac{1}{8} + \tfrac{35}{64}g^2 + \tfrac{25025}{9216}g^4)J^4 \\ & + (\tfrac{7}{48}g + \tfrac{1001}{1152}g^3)J^5 . \end{aligned} \tag{9.42}$$

Taking the logarithm, we find

$$\begin{aligned} W(g,J) = {} & (\tfrac{5}{24}g^2 + \tfrac{5}{16}g^4) + (\tfrac{1}{2}g + \tfrac{5}{8}g^3)J + (\tfrac{1}{2} + \tfrac{1}{2}g^2 + \tfrac{25}{16}g^4)J^2 \\ & + (\tfrac{1}{6}g + \tfrac{2}{3}g^3)J^3 + (\tfrac{1}{8}g^2 + g^4)J^4 + \tfrac{1}{8}g^3J^5 . \end{aligned} \tag{9.43}$$

It is straightforward to check that the symmetry factors given for the diagrams satisfy the sum rule.

c) This follows immediately from the discussion of tadpole cancellation in the text.

d) From eq. (9.43), we have

$$\begin{aligned} \frac{\partial}{\partial J} W(g, J{+}Y)\Big|_{J=0} = {} & (\tfrac{1}{2}g + \tfrac{5}{8}g^3) + (1 + g^2 + \tfrac{25}{8}g^4)Y + (\tfrac{1}{2}g + 2g^3)Y^2 \\ & + (\tfrac{1}{2}g^2 + 4g^4)Y^3 + \tfrac{5}{8}g^3Y^4 . \end{aligned} \tag{9.44}$$

Setting $Y = a_1 g + a_3 g^3$ and setting the result equal to zero, we get

$$0 = (\tfrac{1}{2} + a_1)g + (\tfrac{5}{8} + a_1 + \tfrac{1}{2}a_1^2 + a_3)g^3 + O(g^5). \tag{9.45}$$

The solution is $a_1 = -\frac{1}{2}$ and $a_3 = -\frac{1}{4}$. Setting $Y = -\frac{1}{2}g - \frac{1}{4}g^3$, we get

$$\begin{aligned} W(g, J{+}Y) = {} & (\tfrac{1}{12}g^2 + \tfrac{5}{48}g^4) + (\tfrac{1}{2} + \tfrac{1}{4}g^2 + \tfrac{5}{8}g^4)J^2 \\ & + (\tfrac{1}{6}g + \tfrac{5}{12}g^3)J^3 + (\tfrac{1}{8}g^2 + \tfrac{11}{16}g^4)J^4 , \end{aligned} \tag{9.46}$$

where we have dropped the J^5 term, since it receives a contribution from the uncomputed J^6 terms in $W(g, J)$. It is straightforward to check that the symmetry factors given for the diagrams without tadpoles satisfy the sum rule.

9.5) a) Follows implicitly from the analysis in section 3, but can be shown directly by writing the time derivatives as commutators with H_0, and working them out.

b) Follows immediately from eqs. (9.33) and (9.34). $U(t)$ is unitary because it is the product of two manifestly unitary operators.

c) $i\frac{d}{dt}U(t) = e^{iH_0t}(-H_0+H)e^{-iHt} = e^{iH_0t}H_1e^{-iHt} = (e^{iH_0t}H_1e^{-iH_0t})(e^{iH_0t}e^{-iHt}) = H_I(t)U(t)$, and it is obvious that $U(0) = 1$.

d) Consider, e.g., $\mathcal{H}_1 \propto \varphi(\mathbf{x}, 0)^n$. Then $\mathcal{H}_I(t) \propto e^{iH_0t}\varphi(\mathbf{x}, 0)^n e^{-iH_0t}$. We can insert a factor of $1 = e^{-iH_0t}e^{iH_0t}$ between each pair of fields, and then use eq. (9.34) to get $\mathcal{H}_I(t) \propto \varphi_I(\mathbf{x}, t)^n$.

e) Differentiating with respect to t inside the time-ordering symbol brings down a factor of $-iH_I(t)$. Since t is the latest time, this factor of $-iH_I(t)$ is placed at the far left, QED. For $t < 0$, we must use *anti time ordering*, where operators at later times are placed to the *right* of those at earlier times.

f) Hermitian conjugation of a time-ordered product gives one that is anti time ordered. Then the anti-time-ordered terms in $U^\dagger(t_1)$ cancel those in $U(t_2)$, leaving eq. (9.35). If $t_1 > t_2$, then we must use anti time ordering.

g) $U^\dagger(t_2, t_1) = U(t_2, t_1)$ follows immediately from the definition of $U(t_1, t_2)$. $U(t_3, t_1) = U(t_3, t_2)U(t_2, t_1)$ is obvious from eq. (9.35) if $t_3 > t_2 > t_1$. Otherwise, cancellations between time-ordered and anti-time-ordered terms still yield this result.

h) Follows immediately from $\varphi(x) = U^\dagger(t)\varphi_I(x)U(t)$ and $U(t_2, t_1) = U(t_2)U^\dagger(t_1)$.

i) Follows immediately from part (g).

j) $U(-\infty, 0)|0\rangle = e^{i(1-i\epsilon)H_0(-\infty)}e^{-iH(-\infty)}|0\rangle$, and $e^{-iHt}|0\rangle = |0\rangle$ for any t. Then we write $|0\rangle = \sum_n |n\rangle\langle n|0\rangle$, where the $|n\rangle$'s are the eigenstates of H_0. With $\epsilon > 0$, only the ground state $|\emptyset\rangle$ survives. A similar analysis gives $\langle 0|U^\dagger(\infty, 0) = \langle 0|\emptyset\rangle\langle\emptyset|$.

k) Follows immediately from the results of parts (h), (i), and (j).

l) The U's in part (k), if placed in time order, multiply out to $U(\infty, -\infty)$.

m) Follows immediately from setting every $\varphi(x) = 1$ in eq. (9.39), and using $\langle 0|0\rangle = \langle\emptyset|\emptyset\rangle = 1$.

10 Scattering Amplitudes and the Feynman Rules

10.1) We expand the exponential to second order in $\mathcal{H}_I$, and then compute the correlation function using free-field theory. The second-order term in the numerator of eq. (9.41) is then

$$\tfrac{1}{2}(ig/6)^2 \int d^4y\, d^4z\, \langle 0|\mathrm{T}\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\varphi^3(y)\varphi^3(z)|0\rangle \qquad (10.17)$$

(where we drop the I subscript and the slash through the zero for notational convenience). Here we have set $x_1' = x_3$ and $x_2' = x_4$ to facilitate counting. It is also convenient to write $\varphi^3(y) = \varphi(y_1)\varphi(y_2)\varphi(y_3)$ with $y_i \equiv y$, and similarly for $\varphi^3(z)$, again to facilitate keeping track of terms. We have a total of 10 fields now, and there are $(10-1)!! = 945$ terms on the right-hand side of Wick's theorem. Terms where the x_i's are all paired with each other are canceled by the expansion of the denominator in eq. (9.41). We also drop tadpoles, as per the discussion in section 9, and terms that are not fully connected, as per the discussion in section 10. The terms remaining pair each x_i with a y_i or z_i, and one y_i with one z_i. Given a pairing of this type, there are 3! permutations of the y_i's that yield the same result (after setting $y_i = y$), and 3! permutations of the z_i's that yield the same result (after setting $z_i = z$). Also, two pairings that are identical except for the exchange of y and z yield the same result after these variables are integrated. Pairings differ by whether x_1 is paired with the same dummy variable (y or z) as x_2, or x_3, or x_4. Accounting for all these factors yields eq. (10.9).

10.2) In problem 9.3, we drew "charge" arrows that pointed away from J's and towards $J^\dagger$'s. After taking functional derivatives with respect to J or $J^\dagger$, these arrows will point towards external φ's (and therefore away from the attached vertex) and away from external $\varphi^\dagger$'s (and therefore towards the attached vertex). We saw in problem 5.1 that outgoing a and incoming b particles result in a φ, and that incoming a and outgoing b particles result in a $\varphi^\dagger$. Therefore, incoming a and outgoing b particles correspond to external lines with charge arrows pointed towards the vertex, and outgoing a and incoming b particles correspond to external lines with charge arrows pointed away from the vertex. On the other hand, incoming particles have momentum arrows that point towards the vertex, and outgoing particles have momentum arrows that point away from the vertex. Thus, we can use charge arrows for momenta if we include minus signs with the momenta for incoming and outgoing b particles. Therefore, we have the following Feynman rules (for tree-level processes):

1. For each incoming a particle, draw a line with an arrow pointed towards the vertex, and label it with the a particle's four-momentum, k_i.
2. For each outgoing a particle, draw a line with an arrow pointed away from the vertex, and label it with the a particle's four-momentum, k_i'.
3. For each incoming b particle, draw a line with an arrow pointed away from the vertex, and label it with minus the b particle's four-momentum, $-k_i$.
4. For each outgoing b particle, draw a line with an arrow pointed towards the vertex, and label it with minus the b particle's four-momentum, $-k_i'$.
5. The only allowed vertex joins four lines, two with arrows pointing towards it and two with arrows pointing away from it. Using this vertex, join up all the external lines, including extra internal lines as needed. In this way, draw all possible diagrams that are topologically inequivalent.

6. Assign each internal line its own four-momentum. Think of the four-momenta as flowing along the arrows, and conserve four-momentum at each vertex. For a tree diagram, this fixes the momenta on all the internal lines.
7. The value of a diagram consists of the following factors:
   for each incoming or outgoing particle, 1;
   for each vertex, $-i\lambda$;
   for each internal line, $-i/(k^2 + m^2 - i\epsilon)$.
8. The value of $i\mathcal{T}$ (at tree level) is given by a sum over the values of the contributing diagrams.

10.3) The vertex joins one dashed and two solid lines, with one arrow pointing towards the vertex and one away. The vertex factor is ig.

10.4) Using the method of problem 10.1, the vertex factor for three lines with arrows all pointing towards the vertex can be determined from the free-field theory matrix element

$$\begin{aligned}\langle 0|\varphi\partial^\mu\varphi\partial_\mu\varphi|k_1k_2k_3\rangle &= \partial_2\cdot\partial_3\langle 0|\varphi(x_1)\varphi(x_2)\varphi(x_3)|k_1k_2k_3\rangle\Big|_{x_1=x_2=x_3=x} \\ &= \partial_2\cdot\partial_3\Big[e^{i(k_1x_1+k_2x_2+k_3x_3)} + 5 \text{ perms of } k_i\text{'s}\Big]_{x_1=x_2=x_3=x} \\ &= i^2(2k_2\cdot k_3 + 2k_3\cdot k_1 + 2k_1\cdot k_2)e^{i(k_1+k_2+k_3)x} \; . \end{aligned} \tag{10.18}$$

The vertex factor is then $\frac{1}{2}ig$ times the coefficient of the plane-wave factor on the right-hand side of eq. (10.18). Since $k_1+k_2+k_3 = 0$, we have $(k_1+k_2+k_3)^2 = 0$, and therefore the factor in parentheses on the the right-hand side of eq. (10.18) can be rewritten as $-(k_1^2 + k_2^2 + k_3^2)$. The overall vertex factor, for three lines with arrows all pointing towards the vertex, is then $\frac{1}{2}ig(k_1^2 + k_2^2 + k_3^2)$.

10.5) We take $\varphi \to \varphi + \lambda\varphi^2$. The lagrangian becomes

$$\begin{aligned}\mathcal{L} &= -\tfrac{1}{2}\partial^\mu(\varphi+\lambda\varphi^2)\partial_\mu(\varphi+\lambda\varphi^2) - \tfrac{1}{2}m^2(\varphi+\lambda\varphi^2)^2 \\ &= -\tfrac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \tfrac{1}{2}m^2\varphi^2 - 2\lambda\varphi\partial^\mu\varphi\partial_\mu\varphi - \lambda m^2\varphi^3 - 2\lambda^2\varphi^2\partial^\mu\varphi\partial_\mu\varphi - \tfrac{1}{2}\lambda^2m^2\varphi^4 \; . \end{aligned} \tag{10.19}$$

Using our results from problem 10.4, the three-point vertex factor is

$$\begin{aligned}\mathbf{V}_3 &= (-2i\lambda)(k_1^2 + k_2^2 + k_3^2) - 6i\lambda m^2 \\ &= (-2i\lambda)[(k_1^2 + m^2) + (k_2^2 + m^2) + (k_3^2 + m^2)] \; , \end{aligned} \tag{10.20}$$

and the four-point vertex factor is

$$\begin{aligned}\mathbf{V}_4 &= (-2i\lambda^2)(2!)(k_1^2 + k_2^2 + k_3^2 + k_4^2) - 12i\lambda m^2 \\ &= (-4i\lambda^2)[(k_1^2 + m^2) + \ldots + (k_4^2 + m^2)] + 4i\lambda^2m^2 \; , \end{aligned} \tag{10.21}$$

where all momentum arrows point towards the vertex. The factor of 2! in the first term in $\mathbf{V}_4$ comes from matching external momenta with the two φ's without derivatives.

Now consider $\varphi\varphi \to \varphi\varphi$ scattering. We have the diagrams of fig. 10.2, plus a four-point vertex. In these diagrams, each three-point vertex connects two on-shell external lines with

$k_i^2 = -m^2$, and one internal line. In the s-channel diagram, the internal line has $k_i^2 = -s$; thus each vertex in this diagram has the value $\mathbf{V}_3 = (-2i\lambda)(-s + m^2)$. For the t- and u-channel diagrams, s is replaced by t or u. In the four-point diagram, all lines are external and on-shell, and so the value of the four-point vertex is $\mathbf{V}_4 = 4i\lambda^2 m^2$. We therefore have

$$\begin{aligned} i\mathcal{T} &= [(-2i\lambda)(-s+m^2)]^2 \, \frac{1}{i} \, \frac{1}{-s+m^2} + (s \to t) + (s \to u) + 4i\lambda^2 m^2 \\ &= 4i\lambda^2[(-s+m^2) + (-t+m^2) + (-u+m^2) + m^2] \\ &= 4i\lambda^2(-s-t-u+4m^2) \\ &= 0 \, . \end{aligned} \tag{10.22}$$

11 Cross Sections and Decay Rates

11.1) a) The vertex factor is $2ig$, and so the tree-level amplitude for $A \to BB$ is simply $\mathcal{T} = 2g$. We use eqs. (11.48) and (11.49) with $n' = 2$, $E_1 = m_A$, and $S = 2$. We also use eq. (11.30) with $s = m_A^2$ and $|\mathbf{k}_1'|$ given by eq. (11.3) with $m_{1'} = m_{2'} = m_B$; this yields $|\mathbf{k}_1'| = \frac{1}{2}(m_A^2 - 4m_B^2)^{1/2}$. The integral over $d\Omega$ simply yields 4π. So we find

$$\Gamma = \frac{g^2}{8\pi m_A}\left(1 - \frac{4m_B^2}{m_A^2}\right)^{1/2} \tag{11.60}$$

at tree level.

b) Everything is the same, except that now the vertex factor is ig rather than $2ig$, and the outgoing particles are not identical (one is an a particle and one is a b particle), so we have $S = 1$ rather than 2. Therefore

$$\Gamma = \frac{g^2}{16\pi m_\varphi}\left(1 - \frac{4m_\chi^2}{m_\varphi^2}\right)^{1/2} \tag{11.61}$$

at tree level.

11.2) a) Let the incoming and outgoing electron four-momenta be p and p', and the incoming and outgoing photon four-momenta be k and k'. In the FT frame, we have

$$\begin{aligned} p &= (m, 0, 0, 0)\ , \\ k &= (\omega, 0, 0, \omega)\ , \\ k' &= (\omega', \omega' \sin\theta, 0, \omega' \cos\theta)\ , \end{aligned} \tag{11.62}$$

where $\theta \equiv \theta_{\rm FT}$; p' is fixed by momentum conservation to be $p' = p + k - k'$. We have

$$\begin{aligned} s &= -(p+k)^2 \\ &= (m+\omega)^2 - \omega^2 \\ &= m^2 + 2m\omega\ , \end{aligned} \tag{11.63}$$

$$\begin{aligned} u &= -(p-k')^2 \\ &= (m-\omega')^2 - \omega'^2 \\ &= m^2 - 2m\omega'\ . \end{aligned} \tag{11.64}$$

b) We get a relation among θ, ω, and ω' by using $p'^2 = -m^2$; this yields

$$\begin{aligned} -m^2 &= (p+k-k')^2 \\ &= p^2 + k^2 + k'^2 + 2p{\cdot}k - 2p{\cdot}k' - 2k{\cdot}k' \\ &= -m^2 + 0 + 0 - 2m\omega + 2m\omega' - 2\omega\omega'(\cos\theta - 1)\ . \end{aligned} \tag{11.65}$$

We thus find

$$1 - \cos\theta = m\left(\frac{1}{\omega'} - \frac{1}{\omega}\right). \tag{11.66}$$

c) We use eqs. (11.63) and (11.64) in eq. (11.50) to get

$$|\mathcal{T}|^2 = 32\pi^2\alpha^2\left[\frac{m^2+m\omega+\omega\omega'}{\omega^2}+\frac{m^2-m\omega'+\omega\omega'}{\omega'^2}-\frac{2m^2+m\omega-m\omega'}{\omega\omega'}\right]. \tag{11.67}$$

Organizing terms by powers of m, we find

$$\begin{aligned}|\mathcal{T}|^2 &= 32\pi^2\alpha^2\left[m^2\left(\frac{1}{\omega^2}+\frac{1}{\omega'^2}-\frac{2}{\omega\omega'}\right)+2m\left(\frac{1}{\omega}-\frac{1}{\omega'}\right)+\frac{\omega}{\omega'}+\frac{\omega'}{\omega}\right]\\ &= 32\pi^2\alpha^2\left[(1-\cos\theta)^2-2(1-\cos\theta)+\frac{\omega}{\omega'}+\frac{\omega'}{\omega}\right]\\ &= 32\pi^2\alpha^2\left[-\sin^2\theta+\frac{\omega}{\omega'}+\frac{\omega'}{\omega}\right]. \end{aligned}\tag{11.68}$$

Now we use eq. (11.34),

$$\frac{d\sigma}{dt}=\frac{1}{64\pi s|\mathbf{k}_1|^2_{\rm CM}}\,|\mathcal{T}|^2\;. \tag{11.69}$$

From eq. (11.9), we have $s|\mathbf{k}_1|^2_{\rm CM}=m^2\omega^2$. We can now get $d\sigma/d\Omega_{\rm FT}$ from $d\sigma/dt$ by computing dt with s (and hence ω) held fixed. Solving eq. (11.66) for ω' yields

$$\omega'=\frac{m\omega}{m+\omega(1-\cos\theta)}\;, \tag{11.70}$$

and so

$$\begin{aligned}d\omega' &= \frac{m\omega^2}{[m+\omega(1-\cos\theta)]^2}\,d\cos\theta\\ &= \frac{\omega'^2}{m}\,d\cos\theta\;. \end{aligned}\tag{11.71}$$

We have $t=2m^2-s-u=2m(\omega'-\omega)$. Therefore,

$$\begin{aligned}dt &= 2m\,d\omega'\\ &= 2\omega'^2\,d\cos\theta\\ &= (\omega'^2/\pi)\,d\Omega_{\rm FT}\;. \end{aligned}\tag{11.72}$$

Thus we have

$$\begin{aligned}\frac{d\sigma}{d\Omega_{\rm FT}} &= \frac{\omega'^2}{\pi}\,\frac{d\sigma}{dt}\\ &= \frac{1}{64\pi^2m^2}\,\frac{\omega'^2}{\omega^2}\,|\mathcal{T}|^2\;. \end{aligned}\tag{11.73}$$

Combining this with eq. (11.68), we get eq. (11.51).

11.3) a) This follows immediately from eq. (11.53) and the definition of $d\mathrm{LIPS}_{n'}(k)$, eq. (11.23).

b) The left-hand side is a tensor with two vector indices, and the only-four vector it can depend on is k. The only tensors with two vector indices that can be built out of a single four-vector are $g^{\mu\nu}$ and $k^\mu k^\nu$, so the right-hand side must be a linear combination of these, with scalar coefficients. By dimensional analysis, A and B are dimensionless. The only scalar quantity that A and B could depend on is $k^2 = -m^2$, but since this is dimensionful, A and B must be pure numbers.

c) For $m_{1'} = m_{2'} = 0$, we have $|\mathbf{k}_1'| = \frac{1}{2}\sqrt{s}$. We then use eq. (11.30); integrating over $d\Omega_{\rm CM}$ yields a factor of 4π.

d) Contracting eq. (11.55) with $g^{\mu\nu}$, we get

$$\int (k_1'\cdot k_2')\, d\text{LIPS}_2(k) = (4A + B)k^2 \ . \tag{11.74}$$

Contracting eq. (11.55) with $k^\mu k^\nu$, we get

$$\int (k\cdot k_1')(k\cdot k_2')\, d\text{LIPS}_2(k) = (A + B)(k^2)^2 \ . \tag{11.75}$$

The delta function in $d\text{LIPS}_2(k)$ enforces $k_1'+k_2' = k$. We also have $k_i'^2 = -m_i'^2 = 0$. Therefore $k\cdot k_1' = (k_1' + k_2')\cdot k_1' = k_1'\cdot k_2'$ and similarly $k\cdot k_2' = k_1'\cdot k_2'$. Also, $k^2 = (k_1' + k_2')^2 = 2k_1'\cdot k_2'$. Therefore $k_1'\cdot k_2' = \frac{1}{2}k^2$ and $(k\cdot k_1')(k\cdot k_2') = \frac{1}{4}(k^2)^2$. Using these in eqs. (11.74) and (11.75), and then using eq. (11.56), we find $4A + B = 1/16\pi$ and $A + B = 1/32\pi$, which yields $A = 1/48\pi$ and $B = 1/96\pi$.

11.4) We have

$$\begin{aligned}
\mathcal{T}_{AA\to AA} &= 0 \ , \\
\mathcal{T}_{AA\to AB} &= 0 \ , \\
\mathcal{T}_{AA\to BB} &= g^2\left[\frac{1}{m_C^2 - t} + \frac{1}{m_C^2 - u}\right], \\
\mathcal{T}_{AA\to BC} &= 0 \ , \\
\mathcal{T}_{AB\to AB} &= g^2\left[\frac{1}{m_C^2 - s} + \frac{1}{m_C^2 - u}\right], \\
\mathcal{T}_{AB\to AC} &= 0 \ ,
\end{aligned} \tag{11.76}$$

12 Dimensional Analysis with $\hbar = c = 1$

12.1) $\hbar c = 0.197327\,\mathrm{GeV\,fm}$.

12.2) $m_p = 0.93827\,\mathrm{GeV}$,
$m_n = 0.93957\,\mathrm{GeV}$,
$m_{\pi^\pm} = 0.13957\,\mathrm{GeV}$,
$m_{\pi^0} = 0.13498\,\mathrm{GeV}$,
$m_e = 0.51100 \times 10^{-3}\,\mathrm{GeV}$,
$m_\mu = 0.10566\,\mathrm{GeV}$,
$m_\tau = 1.7770\,\mathrm{GeV}$.

12.3) By dimensional analysis, r_p must be proportional to $\hbar c/m_p$. The proton is a blob of strongly interacting quarks and gluons, and there is no small dimensionless parameter associated with it; therefore we expect the constant of proportionality to be $O(1)$. Our guestimate is then $r_p \sim \hbar c/m_p = 0.2\,\mathrm{fm}$. The measured value is $0.875\,\mathrm{fm}$.

13 The Lehmann-Källén Form of the Exact Propagator

13.1) We start with eq. (13.12), take $\partial/\partial y^0$, and then set $y^0 = x^0$ to get

$$\begin{aligned}\langle 0|\varphi(x)\dot\varphi(y)|0\rangle &= \int \widetilde{dk}\, ik^0\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} + \int_{4m^2}^{\infty} ds\,\rho(s) \int \widetilde{dk}\, ik^0\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \\ &= \frac{i}{2}\int \frac{d^3k}{(2\pi)^3}\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} + \frac{i}{2}\int_{4m^2}^{\infty} ds\,\rho(s) \int \frac{d^3k}{(2\pi)^3}\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \\ &= \frac{i}{2}\,\delta^3(\mathbf{x}-\mathbf{y})\left[1 + \int_{4m^2}^{\infty} ds\,\rho(s)\right] . \end{aligned} \tag{13.19}$$

Similarly, we take $\partial/\partial y^0$ of eq. (13.13) and set $y^0 = x^0$ to get

$$\langle 0|\dot\varphi(y)\varphi(x)|0\rangle = -\frac{i}{2}\,\delta^3(\mathbf{x}-\mathbf{y})\left[1 + \int_{4m^2}^{\infty} ds\,\rho(s)\right] . \tag{13.20}$$

Subtracting eq. (13.20) from eq. (13.19), we find

$$\langle 0|[\varphi(x),\dot\varphi(y)]|0\rangle = i\,\delta^3(\mathbf{x}-\mathbf{y})\left[1 + \int_{4m^2}^{\infty} ds\,\rho(s)\right] \tag{13.21}$$

when $y^0 = x^0$. On the other hand, the conjugate momentum to the field φ is $\Pi = \partial\mathcal{L}/\partial\dot\varphi = Z_\varphi\dot\varphi$. The canonical commutation relations tells us that $[\varphi(x),\Pi(y)] = i\delta^3(\mathbf{x}-\mathbf{y})$ when $y^0 = x^0$, and therefore that $Z_\varphi[\varphi(x),\dot\varphi(y)] = i\delta^3(\mathbf{x}-\mathbf{y})$ when $y^0 = x^0$. Comparing with eq. (13.21), we see that we have

$$Z_\varphi^{-1} = 1 + \int_{4m^2}^{\infty} ds\,\rho(s) \; . \tag{13.22}$$

Note that this implies that $Z_\varphi \le 1$, and that $Z_\varphi = 1$ only if $\rho(s) = 0$.

14 Loop Corrections to the Propagator

14.1) Starting with eq. (14.50), we have

$$\frac{\prod_i \Gamma(\alpha_i)}{\prod_i A_i^{\alpha_i}} = \int_0^\infty dt_1 \ldots dt_n \, \prod_i t_i^{\alpha_i - 1} \exp\Bigl(-\textstyle\sum_i A_i t_i\Bigr) \,. \tag{14.55}$$

Now we insert a factor of $1 = \int_0^\infty ds \, \delta(s - \sum_i t_i)$ on the right-hand side, and then make the change of variable $t_i = s x_i$. The delta function becomes $\delta(s - s\sum_i x_i) = s^{-1}\delta(1 - \sum_i x_i)$. Then we have

$$\frac{\prod_i \Gamma(\alpha_i)}{\prod_i A_i^{\alpha_i}} = \int_0^\infty dx_1 \ldots dx_n \, \delta(1 - \textstyle\sum_i x_i) \displaystyle\prod_i x_i^{\alpha_i - 1} \int_0^\infty ds \, s^{-1+\sum_i \alpha_i} \exp\Bigl(-s\textstyle\sum_i A_i x_i\Bigr) \,. \tag{14.56}$$

The integral over s yields $\Gamma(\sum_i \alpha_i)(\sum_i A_i x_i)^{-\sum_i \alpha_i}$. The integrals over the x_i's, along with the delta function, constitute the integral over dF_n divided by $(n{-}1)!$. So we have

$$\frac{\prod_i \Gamma(\alpha_i)}{\prod_i A_i^{\alpha_i}} = \frac{\Gamma(\sum_i \alpha_i)}{(n{-}1)!} \int dF_n \, \frac{\prod_i x_i^{\alpha_i - 1}}{(\sum_i x_i A_i)^{\sum_i \alpha_i}} \,, \tag{14.57}$$

which is equivalent to eq. (14.49).

14.2) We define $I_d \equiv \int d^d x \, e^{-\mathbf{x}^2}$. In cartesian coordinates, $I_d = \prod_{i=1}^d \int_{-\infty}^{+\infty} dx_i \, e^{-x_i^2} = (\sqrt{\pi})^d = \pi^{d/2}$. In spherical coordinates, $I_d = \Omega_d \int_0^\infty dr \, r^{d-1} \, e^{-r^2}$. Let $u = r^2$; then $r\,dr = \frac{1}{2}du$, and we have $I_d = \frac{1}{2}\Omega_d \int_0^\infty du \, u^{d/2-1} \, e^{-u} = \frac{1}{2}\Omega_d \Gamma(\frac{1}{2}d)$.

14.3) a) The integrand in eq. (14.52) is odd under $q \to -q$, and so vanishes when integrated. The left-hand side of eq. (14.53) is a two-index constant symmetric tensor, and so must equal $g^{\mu\nu}A$, where A is a Lorentz scalar. To determine A, we contract both sides with $g_{\mu\nu}$; since $g_{\mu\nu}g^{\mu\nu} = \delta_\mu{}^\mu = d$, we find $C_2 = 1/d$.

b) The result is a constant four-index completely symmetric tensor, and hence must equal $(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\sigma\nu} + g^{\mu\sigma}g^{\nu\rho})B$. Contracting this with $g_{\mu\nu}g_{\rho\sigma}$, we get $(d^2 + d + d)B = d(d{+}2)B$. Therefore

$$\int d^d q \; q^\mu q^\nu q^\rho q^\sigma f(q^2) = \frac{1}{d(d{+}2)}(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\sigma\nu} + g^{\mu\sigma}g^{\nu\rho}) \int d^d q \; (q^2)^2 f(q^2) \,. \tag{14.58}$$

14.4) We subtract eq. (14.43) from eq. (14.39), divide by α, and drop higher-order terms to get

$$\begin{aligned} 0 &= \tfrac{1}{2} \int_0^1 dx \, D \ln(D_0/m^2) + (\tfrac{1}{6}\kappa_A + \tfrac{1}{12})k^2 + (\kappa_B + \tfrac{1}{12})m^2 \\ &= \tfrac{1}{36}(3\pi\sqrt{3} - 17)k^2 + \tfrac{1}{6}(\pi\sqrt{3} - 6)m^2 + (\tfrac{1}{6}\kappa_A + \tfrac{1}{12})k^2 + (\kappa_B + \tfrac{1}{12})m^2 \,. \end{aligned} \tag{14.59}$$

So we find $\kappa_A = \frac{1}{6}(14 - 3\pi\sqrt{3}) = -0.3874$ and $\kappa_B = \frac{1}{12}(11 - 2\pi\sqrt{3}) = 0.0098$.

14.5) See section 31. From eq. (31.5), we have

$$\Pi(k^2) = \frac{\lambda}{16\pi^2}\biggl[\frac{1}{\varepsilon} + \tfrac{1}{2} + \ln(\mu/m)\biggr]m^2 - Ak^2 - Bm^2 + O(\lambda^2) \,. \tag{14.60}$$

We see immediately that

$$\begin{aligned} A &= O(\lambda^2)\ , \\ B &= \frac{\lambda}{16\pi^2}\left[\frac{1}{\varepsilon}+\tfrac{1}{2}+\ln(\mu/m)\right]+O(\lambda^2)\ . \end{aligned} \tag{14.61}$$

and that $\Pi(k^2)=0$ to $O(\lambda)$.

14.6) The only difference is that the loop has a symmetry factor of $S=1$ rather than $S=2$, so there is an extra factor of 2 in the loop correction, and hence in B; A is still zero at one loop.

14.7) a) Set $Q=(2\omega)^{-1/2}(a^\dagger+a)$ and $P=i(\omega/2)^{1/2}(a^\dagger-a)$, where $[a,a^\dagger]=1$. Then $H_0=\omega(a^\dagger a+\frac{1}{2})$, and

$$\begin{aligned} H_1 &= \tfrac{1}{2}(Z^{-1}{-}1)P^2+\tfrac{1}{2}(Z_\omega{-}1)\omega^2Q^2+\lambda Q^4 \\ &= \lambda\Big[-\tfrac{1}{2}\kappa_A P^2+\tfrac{1}{2}\kappa_B Q^2+Q^4\Big]+O(\lambda^2)\ . \\ &= \tfrac{1}{4}\lambda\omega\Big[\kappa_A(a^\dagger{-}a)^2+\kappa_B(a^\dagger{+}a)^2+(a^\dagger{+}a)^4\Big]+O(\lambda^2)\ . \end{aligned} \tag{14.62}$$

b) Using $a|n\rangle=\sqrt{n}\,|n{-}1\rangle$, $a^\dagger|n\rangle=\sqrt{n{+}1}\,|n{+}1\rangle$, and $a^\dagger a|n\rangle=n|n\rangle$, we find $\langle 1|Q|0\rangle=(2\omega)^{-1/2}$, and

$$\begin{aligned} \langle n'|(a^\dagger{-}a)^2|n\rangle = {} & \sqrt{(n{+}2)(n{+}1)}\,\delta_{n',n+2} \\ & -(2n{+}1)\,\delta_{n',n} \\ & +\sqrt{n(n{-}1)}\,\delta_{n',n-2}\ , \end{aligned} \tag{14.63}$$

$$\begin{aligned} \langle n'|(a^\dagger{+}a)^2|n\rangle = {} & \sqrt{(n{+}2)(n{+}1)}\,\delta_{n',n+2} \\ & +(2n{+}1)\,\delta_{n',n} \\ & +\sqrt{n(n{-}1)}\,\delta_{n',n-2}\ , \end{aligned} \tag{14.64}$$

$$\begin{aligned} \langle n'|(a^\dagger{+}a)^4|n\rangle = {} & \sqrt{(n{+}4)(n{+}3)(n{+}2)(n{+}1)}\,\delta_{n',n+4} \\ & +(4n{+}6)\sqrt{(n{+}2)(n{+}1)}\,\delta_{n',n+2} \\ & +(6n^2{+}6n{+}3)\,\delta_{n',n} \\ & +(4n{-}2)\sqrt{n(n{-}1)}\,\delta_{n',n+2} \\ & +\sqrt{n(n{-}1)(n{-}2)(n{-}3)}\,\delta_{n',n+4}\ . \end{aligned} \tag{14.65}$$

We have in general that $E_N=\varepsilon_n+\langle n|H_1|n\rangle+O(\lambda^2)$, where $\varepsilon_n=(n{+}\frac{1}{2})\omega$ is the unperturbed energy, and so

$$E_\Omega = \tfrac{1}{2}\omega+\tfrac{1}{4}\lambda\omega(-\kappa_A+\kappa_B+3)+O(\lambda^2)\ , \tag{14.66}$$

$$E_{\rm I} = \tfrac{3}{2}\omega+\tfrac{1}{4}\lambda\omega(-3\kappa_A+3\kappa_B+15)+O(\lambda^2)\ . \tag{14.67}$$

For the states, we have in general that

$$|N\rangle = |n\rangle + \sum_{n'\neq n} \frac{\langle n'|H_1|n\rangle}{\varepsilon_{n'} - \varepsilon_n}\, |n'\rangle + O(\lambda^2)\;, \tag{14.68}$$

and so

$$|\Omega\rangle \,=\, |0\rangle + \tfrac{1}{4}\lambda\left[\frac{\kappa_A\sqrt{2} + \kappa_B\sqrt{2} + 6\sqrt{2}}{2}\,|2\rangle + \frac{\sqrt{24}}{4}\,|4\rangle\right] + O(\lambda^2)\;, \tag{14.69}$$

$$|\mathrm{I}\rangle \,=\, |1\rangle + \tfrac{1}{4}\lambda\left[\frac{\kappa_A\sqrt{6} + \kappa_B\sqrt{6} + 10\sqrt{6}}{2}\,|3\rangle + \frac{\sqrt{120}}{4}\,|5\rangle\right] + O(\lambda^2)\;. \tag{14.70}$$

c) From eqs. (14.66) and (14.67), we see that requiring $E_{\mathrm{I}} - E_\Omega \equiv \omega$ fixes $\kappa_A - \kappa_B = 6$. Next, we act on $|\Omega\rangle$ with $\sqrt{2\omega}\,Q = a^\dagger + a$; from eq. (14.69), we find

$$\begin{aligned}\sqrt{2\omega}\,Q|\Omega\rangle \,=\, |1\rangle + \tfrac{1}{4}\lambda\Bigg[&\frac{\kappa_A\sqrt{2} + \kappa_B\sqrt{2} + 6\sqrt{2}}{2}\Big(\sqrt{3}\,|3\rangle + \sqrt{2}\,|1\rangle\Big) \\ &+ \frac{\sqrt{24}}{4}\Big(\sqrt{5}\,|5\rangle + \sqrt{4}\,|3\rangle\Big)\Bigg] + O(\lambda^2)\;.\end{aligned} \tag{14.71}$$

Using eq. (14.70), we find that

$$\sqrt{2\omega}\,\langle \mathrm{I}|Q|\Omega\rangle = 1 + \tfrac{1}{4}\lambda(\kappa_A + \kappa_B + 6) + O(\lambda^2)\;. \tag{14.72}$$

Requiring $\sqrt{2\omega}\,\langle \mathrm{I}|Q|\Omega\rangle \equiv 1$ fixes $\kappa_A + \kappa_B = -6$. Hence $\kappa_A = 0$ and $\kappa_B = -6$.

d) Using

$$i\Pi(k^2) = \tfrac{1}{2}(-i\lambda\tilde{\mu}^{\varepsilon})\tfrac{1}{i}\tilde{\Delta}(0) - i(Ak^2 + Bm^2) \tag{14.73}$$

from section 30, with the substitutions $m \to \omega$ and $\lambda \to 24\lambda\omega^3$, we find that the self-energy is

$$i\Pi(k^2) = \tfrac{1}{2}(-24i\lambda\omega^3)\tfrac{1}{i}\tilde{\Delta}(0) - i\lambda(\kappa_A k^2 + \kappa_B\omega^2) + O(\lambda^2)\;, \tag{14.74}$$

where, after making the Wick rotation,

$$\begin{aligned}\tilde{\Delta}(0) \,&=\, i\int_{-\infty}^{+\infty} \frac{d\ell}{2\pi}\,\frac{1}{\ell^2 + \omega^2} \\ &= \frac{i}{2\omega}\;.\end{aligned} \tag{14.75}$$

Requiring $\Pi'(-\omega^2) = 0$ fixes $\kappa_A = 0$, and then requiring $\Pi(-\omega^2) = 0$ fixes $\kappa_B = -6$, as we found in part (c). Agreement is required, as the conditions defining ω and the normalization of Q are the same.

15 The One-Loop Correction in Lehmann-Källén Form

15.1) a) We have $\Pi(k^2) = \Pi_{\rm loop}(k^2) - Ak^2 - Bm^2 + O(\alpha^2)$, and we fix A by requiring $\Pi'(-m^2) = 0$. To $O(\alpha)$, this condition yields $A = \Pi'_{\rm loop}(-m^2)$. Eq. (15.15) then follows immediately from Cauchy's integral formula; see e.g. http://mathworld.wolfram.com/CauchyIntegralFormula.html .

b) The right-hand side of eq. (14.32) is manifestly real if D is real and positive. As discussed in the text, D can be negative only for $k^2 < -4m^2$; then the fractional power $D^{\varepsilon/2}$ results in a branch point.

c) At large $|w|$, $\Pi_{\rm loop}(w)/(w + m^2)^2 \sim |w|^{-1-\varepsilon/2}$, and so along an arc at $|w| = R$ the line integral becomes $R^{-\varepsilon/2}d\theta$, which vanishes as $R \to \infty$.

d) Above the cut, we have $D = e^{i(\pi-\epsilon)}|D|$, and below the cut, we have $D = e^{-i(\pi-\epsilon)}|D|$, where ϵ (not to be confused with ε!) is a positive infinitesimal. Thus $D^{\varepsilon/2} = e^{i\pi\varepsilon/2}|D|$ above the cut, and $D^{\varepsilon/2} = e^{-i\pi\varepsilon/2}|D|$ below the cut (where we have now taken ϵ to zero). We see that the real parts match, and the imaginary parts have opposite sign; this implies eq. (15.17).

e) Eq. (15.18) follows immediately from eqs. (15.16) and (15.17). Note that there are three overall minus signs: one from eq. (15.16), one from $ds = -dw$, and one from swapping the limits of integration. Examining eq. (15.13), we see that the integrand in eq. (15.18) is simply the $O(\alpha)$ contribution to $\pi\rho(s)$. [The $s \to s + i\epsilon$ prescription is implicit in eq. (15.18).] So we conclude that, to $O(\alpha)$, $A = -\int_{4m^2}^{\infty} ds\, \rho(s)$. Since $Z_\varphi^{-1} = 1 - A + O(\alpha^2)$, this verifies $Z_\varphi^{-1} = 1 + \int_{4m^2}^{\infty} ds\, \rho(s)$ to $O(\alpha)$.

Incidentally, you could try to carry out this integral over s with finite ε, then take the $\varepsilon \to 0$ limit, and hence get the value of κ_A. Doing this with Mathematica yields some horrible expression in terms of hypergeometric functions for κ_A, but numerically it does agree with our result in problem 14.4.

15.2) We start with the Cauchy integral formula for second derivative,

$$\Pi''(k^2) = 2! \oint \frac{dw}{2\pi i} \frac{\Pi(w)}{(w - k^2)^3} \,, \tag{15.20}$$

and follow the analysis above. The only difference is an extra minus sign from the denominator, and the result is eq. (15.19).

16 Loop Corrections to the Vertex

16.1) See section 31. From eq. (31.8), we have

$$\mathbf{V}_4 = -(1+C)\lambda + \tfrac{1}{2}\lambda^2[F(-s)+F(-t)+F(-u)] + O(\lambda^3)\ , \tag{16.16}$$

where

$$F(k^2) = \frac{1}{16\pi^2}\left[\frac{2}{\varepsilon} + \int_0^1 dx\ \ln(\mu^2/D)\right]\ , \tag{16.17}$$

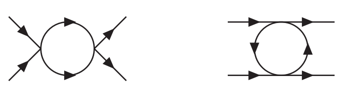
and $D = x(1-x)k^2 + m^2 - i\varepsilon$. We require $\mathbf{V}_4 = -\lambda$ for $s = 4m^2$ and $t = u = 0$; We find

$$\begin{aligned} F(0) &= \frac{1}{16\pi^2}\left[\frac{2}{\varepsilon} + \ln(\mu^2/m^2)\right]\ , \\ F(-4m^2) &= \frac{1}{16\pi^2}\left[\frac{2}{\varepsilon} + \ln(\mu^2/m^2) + 2\right]\ , \end{aligned} \tag{16.18}$$

and so

$$C = \frac{3\lambda}{16\pi^2}\left[\frac{1}{\varepsilon} + \ln(\mu/m) + \tfrac{1}{3}\right] + O(\lambda^2)\ . \tag{16.19}$$

16.2) We must be careful with symmetry factors. For $aa \to aa$, the contributing one-loop diagrams are


+ crossed

The s-channel diagram has a symmetry factor of $S = 2$, while the t- and u-channel diagrams have $S = 1$. So instead of eq. (16.16), we have

$$\mathbf{V}_4 = -(1+C)\lambda + \lambda^2[\tfrac{1}{2}F(-s)+F(-t)+F(-u)] + O(\lambda^3)\ , \tag{16.20}$$

which results in

$$C = \frac{5\lambda}{16\pi^2}\left[\frac{1}{\varepsilon} + \ln(\mu/m) + \tfrac{1}{5}\right] + O(\lambda^2)\ . \tag{16.21}$$

17 Other 1PI Vertices

17.1) We have

$$\begin{aligned}
x_1(\ell{-}k_1)^2 + x_2(\ell{+}k_2)^2 + x_3(\ell{+}k_2{+}k_3)^2 + x_4\ell^2 \\
&= \ell^2 + 2(-x_1k_1{+}x_2k_2{+}x_3(k_2{+}k_3))\ell + x_1k_1^2 + x_2k_2^2 + x_3(k_2{+}k_3)^2 \\
&= [\ell + (-x_1k_1{+}x_2k_2{+}x_3(k_2{+}k_3))]^2 \\
&\quad - (-x_1k_1{+}x_2k_2{+}x_3(k_2{+}k_3))^2 + x_1k_1^2 + x_2k_2^2 + x_3(k_2{+}k_3)^2 \\
&\equiv q^2 + D \,, \qquad (17.6)
\end{aligned}$$

where

$$\begin{aligned}
D &= -(-x_1k_1{+}x_2k_2{+}x_3(k_2{+}k_3))^2 + x_1k_1^2 + x_2k_2^2 + x_3(k_2{+}k_3)^2 \\
&= x_1(1{-}x_1)k_1^2 + x_2(1{-}x_2)k_2^2 + x_3(1{-}x_3)(k_2{+}k_3)^2 \\
&\quad + 2x_1x_2k_1k_2 + 2x_1x_3k_1(k_2{+}k_3) - 2x_2x_3k_2(k_2{+}k_3) \,. \qquad (17.7)
\end{aligned}$$

Next we use

$$2k_1k_2 = (k_1{+}k_2)^2 - k_1^2 - k_2^2 \,, \qquad (17.8)$$

$$\begin{aligned}
2k_1(k_2{+}k_3) &= 2k_1(-k_1{-}k_4) \\
&= -(k_1{+}k_4)^2 - k_1^2 + k_4^2 \\
&= -(k_2{+}k_3)^2 - k_1^2 + k_4^2 \,, \qquad (17.9)
\end{aligned}$$

$$-2k_2(k_2{+}k_3) = -(k_2{+}k_3)^2 - k_2^2 + k_3^2 \qquad (17.10)$$

to get

$$\begin{aligned}
D &= x_1(1{-}x_1{-}x_2{-}x_3)k_1^2 + x_2(1{-}x_2{-}x_1{-}x_3)k_2^2 + x_3(1{-}x_3{-}x_1{-}x_2)(k_2{+}k_3)^2 \\
&\quad + x_1x_2(k_1{+}k_2)^2 + x_1x_3k_4^2 + x_2x_3k_3^2 \\
&= x_1x_4k_1^2 + x_2x_4k_2^2 + x_3x_4(k_2{+}k_3)^2 + x_1x_2(k_1{+}k_2)^2 + x_1x_3k_4^2 + x_2x_3k_3^2 \,, \qquad (17.11)
\end{aligned}$$

QED.

18 Higher-Order Corrections and Renormalizability

18.1) a) $[\mathcal{L}] = d$ and $[\gamma^\mu \partial_\mu] = 1$, so $[\Psi] = [\overline{\Psi}] = \frac{1}{2}(d-1)$.

b) $[g_n] + 2n[\Psi] = d$, so $[g_n] = d - n(d-1)$.

c) $[g_{m,n}] + m[\varphi] + 2n[\Psi] = d$, and $[\varphi] = \frac{1}{2}(d-2)$, so $[g_{m,n}] = d - \frac{1}{2}m(d-2) - n(d-1)$.

d) $[g_{m,n}] = 4 - m - 3n$, so only $[g_{1,1}] = 0$, and all the rest are negative; thus $g_{1,1}\varphi\overline{\Psi}\Psi$ is the only allowed interaction of this type for $d = 4$.

19 Perturbation Theory to All Orders

20 Two-Particle Elastic Scattering at One Loop

20.1) For $m^2 = 0$, we have $-D_4 = 1/(x_1x_2s + x_3x_4t)$; we treat s and t as complex, and so ignore any poles in the Feynman parameter integrand. We then have

$$\begin{aligned}\int \frac{dF_4}{D_4} &= -3!\int_0^1 dx_1 \int_0^{1-x_1} dx_3 \int_0^{1-x_1-x_3} \frac{dx_2}{x_1x_2s + x_3(1{-}x_1{-}x_2{-}x_3)t} \\ &= -3!\int_0^1 dx_3 \int_0^{1-x_3} dx_1\, \frac{\ln s - \ln t + \ln x_1 - \ln x_3}{x_1 s - x_3 t} \\ &= -3!\int_0^1 dx_3 \int_0^{1-x_3} \frac{dx_1}{x_3}\, \frac{\ln(s/t) + \ln(x_1/x_3)}{(x_1/x_3)s - t}\ . \end{aligned} \tag{20.20}$$

Now let $x_1 = yx_3$; $x_1 < 1 - x_3 \Rightarrow y < 1/x_3 - 1$, and so

$$\int \frac{dF_4}{D_4} = -3!\int_0^1 dx_3 \int_0^{1/x_3-1} dy\, \frac{\ln(s/t) + \ln y}{ys - t}\ . \tag{20.21}$$

Note that $y < 1/x_3 - 1$ is equivalent to $x_3 < 1/(1{+}y)$, so we can write

$$\begin{aligned}\int \frac{dF_4}{D_4} &= -3!\int_0^\infty dy \int_0^{1/(1+y)} dx_3\, \frac{\ln(s/t) + \ln y}{ys - t} \\ &= -3!\int_0^\infty dy\, \frac{1}{1+y}\, \frac{\ln(s/t) + \ln y}{ys - t}\ . \end{aligned} \tag{20.22}$$

These are standard integrals:

$$\begin{aligned}\int_0^\infty \frac{dy}{(y+1)(ys-t)} &= \frac{\ln(-s/t)}{s+t}\ , \\ \int_0^\infty \frac{dy\, \ln y}{(y+1)(ys-t)} &= -\frac{[\ln(-s/t)]^2}{2(s+t)}\ . \end{aligned} \tag{20.23}$$

Using $\ln(-s/t) = \ln(s/t) - i\pi$ then yields eq. (20.17).

20.2) From eq. (20.2) with $s = 4m^2$ and $t = u = 0$, we find

$$\mathcal{T}_{1-\text{loop}} = \mathbf{V}_3^2(4m^2)\tilde{\mathbf{\Delta}}(-4m^2) + 2\mathbf{V}_3^2(0)\tilde{\mathbf{\Delta}}(0) + \mathbf{V}_4(4m^2,0,0)\ . \tag{20.24}$$

So we just have to evaluate some Feynman parameter integrals. I will give only the results, computed by Sam Pinansky:

$$\begin{aligned}\tilde{\mathbf{\Delta}}(0) &= \frac{1}{m^2}\Big[1 + \tfrac{1}{12}(11 - 2\pi\sqrt{3})\alpha\Big]\ , \\ \tilde{\mathbf{\Delta}}(-4m^2) &= -\frac{1}{3m^2}\Big[1 + \tfrac{1}{36}(9 - 2\pi\sqrt{3})\alpha\Big]\ , \\ \mathbf{V}_3(0) &= g\Big[1 + \tfrac{1}{6}(6 - \pi\sqrt{3})\alpha\Big]\ , \\ \mathbf{V}_3(4m^2) &= g\Big[1 + \tfrac{1}{6}(8 - \pi\sqrt{3})\alpha\Big]\ , \\ \mathbf{V}_4(4m^2,0,0) &= -\frac{g^2}{m^2}\Big[\tfrac{1}{9}(3 - 2\pi\sqrt{3})\alpha\Big]\ , \end{aligned} \tag{20.25}$$

which yield

$$\mathcal{T}_{1-\text{loop}} = \frac{5g^2}{3m^2}\Big[1 + \tfrac{1}{180}(489 - 70\pi\sqrt{3})\alpha\Big]\ . \tag{20.26}$$

21 The Quantum Action

21.1) This follows immediately from eqs. (21.12), (21.14), and (21.18).

21.2) a) If we make the change of integration variable specified by eq. (21.22) in eq. (21.2), we find that it is equivalent to the replacement of eq. (21.23).

b) For notational convenience, treat the spacetime argument as part of the index. The solution to $\delta W/\delta J_a = \varphi_a$ is called $J_{\varphi a}$. If we now take $\varphi_a \to R_{ab}\varphi_b$, the solution is $J_{\varphi b}(R^{-1})_{ba}$. To see this, let $K_{\varphi a} \equiv J_{\varphi b}(R^{-1})_{ba}$, so that $J_{\varphi b} = K_a R_{ab}$, and compute $\delta W/\delta K_a = (\delta W/\delta J_b)(\delta J_b/\delta K_a) = (\delta W/\delta J_b)R_{ab} = \varphi_b R_{ab} = R_{ab}\varphi_b$. So we let $\varphi_a \to R_{ab}\varphi_b$ and $J_{\varphi a} \to J_{\varphi c}(R^{-1})_{ca}$ in eq. (21.20). The R matrices cancel out of the $J_\varphi \varphi$ term, and we saw in part (a) that $W(J_\varphi)$ is invariant. So $\Gamma(\varphi)$ is also invariant.

21.3) If we make the change of integration variable $\varphi \to \varphi - \bar\varphi$ in eq. (21.24), we find

$$W(J;\bar\varphi) = W(J;0) - \int d^dx\, J\bar\varphi\ . \tag{21.28}$$

Taking $\delta/\delta J(x)$ we find

$$\frac{\delta W(J;\bar\varphi)}{\delta J(x)} = \frac{\delta W(J;0)}{\delta J(x)} - \bar\varphi(x)\ . \tag{21.29}$$

We use eq. (21.26) to identify the left-hand side of eq. (21.29) as $\varphi(x)$, and rearrange to get

$$\frac{\delta W(J;0)}{\delta J(x)} = \varphi(x) + \bar\varphi(x)\ . \tag{21.30}$$

Let $J_{\varphi;\bar\varphi}$ be the solution of eq. (21.26). (It was called J_φ in the problem, but this notation is more useful.) In this notation, the solution of eq. (21.30) is $J_{\varphi+\bar\varphi;0}$. Since eq. (21.30) is equivalent to eq. (21.26), we see that

$$J_{\varphi;\bar\varphi} = J_{\varphi+\bar\varphi;0}\ . \tag{21.31}$$

Starting with eq. (21.25), we find

$$\begin{aligned}\Gamma(\varphi;\bar\varphi) &= W(J_{\varphi;\bar\varphi};\bar\varphi) - \int d^dx\, J_{\varphi;\bar\varphi}\varphi \\ &= W(J_{\varphi;\bar\varphi};0) - \int d^dx\, J_{\varphi;\bar\varphi}(\varphi+\bar\varphi) \\ &= W(J_{\varphi+\bar\varphi;0};0) - \int d^dx\, J_{\varphi+\bar\varphi;0}(\varphi+\bar\varphi) \\ &= \Gamma(\varphi{+}\bar\varphi;0)\ ,\end{aligned} \tag{21.32}$$

where the second equality follows from eq. (21.28), the third from eq. (21.31), and the fourth from eq. (21.20).

22 Continuous Symmetries and Conserved Currents

22.1) In eq. (22.6) with $\mu = 0$, we have $\partial\mathcal{L}/\partial(\partial_0\varphi_a) = \partial\mathcal{L}/\partial\dot\varphi_a = \Pi_a$, and so $j^0 = \Pi_a\delta\varphi_a$. Thus, for $y^0 = x^0$, we have $[\varphi_a(x), j^0(y)] = [\varphi_a(x), \Pi_b(y)]\delta\varphi_b(y) = i\delta^3(\mathbf{x}-\mathbf{y})\delta_{ab}\delta\varphi_b(y)$. Integrating over d^4y yields $[\varphi_a(x), Q]$ on the left-hand side and $i\delta\varphi_b(x)$ on the right-hand side. Since Q is time independent, our choice of $y^0 = x^0$ is justified.

22.2) Taking $y^0 = x^0$, we have $[\varphi_a(x), T^{0i}(y)] = -[\varphi_a(x), \Pi_b(y)]\nabla^i\varphi_b(y) = -i\delta^3(\mathbf{x}-\mathbf{y})\delta_{ab}\nabla^i\varphi_b(y)$. Integrating over d^4y yields $[\varphi_a(x), P^i]$ on the left-hand side and $-i\nabla^i\varphi_a(x)$ on the right-hand side. Since P^i is time independent, our choice of $y^0 = x^0$ is justified. We also have $[\varphi_a(x), T^{00}(y)] = [\varphi_a(x), \frac{1}{2}\Pi_b(y)\Pi_b(y)] = \frac{1}{2}[\varphi_a(x), \Pi_b(y)]\Pi_b(y) + \frac{1}{2}\Pi_b(y)[\varphi_a(x), \Pi_b(y)] = i\delta^3(\mathbf{x}-\mathbf{y})\delta_{ab}\Pi_b(y)$. Integrating over d^4y yields $[\varphi_a(x), P^0]$ on the left-hand side and $i\Pi_a(x) = i\dot\varphi_a(x) = -i\partial^0\varphi_a(x)$ on the right-hand side. Since P^0 is time independent, our choice of $y^0 = x^0$ is justified.

22.3) a) We have, with $y^0 = x^0$,

$$\begin{aligned}
[T^{00}(x), T^{00}(y)] &= \tfrac{1}{2}[\Pi_a(x)\Pi_a(x), \tfrac{1}{2}\nabla^j\varphi_a(y)\nabla^j\varphi_a(y) + V(\varphi(y))] - (x \leftrightarrow y) \\
&= -\tfrac{1}{2}i\Pi_a(x)[\nabla^j\varphi_a(y)\nabla^j_y + \partial V(\varphi)/\partial\varphi_a]\delta^3(\mathbf{x}-\mathbf{y}) \\
&\quad -\tfrac{1}{2}i[\nabla^j\varphi_a(y)\nabla^j_y + \partial V(\varphi)/\partial\varphi_a]\delta^3(\mathbf{x}-\mathbf{y})\Pi_a(x) - (x \leftrightarrow y)\ ,
\end{aligned} \tag{22.42}$$

$$\begin{aligned}
[T^{0i}(x), T^{00}(y)] &= -[\Pi_a(x), \tfrac{1}{2}\nabla^j\varphi_a(y)\nabla^j\varphi_a(y) + V(\varphi(y))]\nabla^i\varphi_a(x) \\
&\quad -\Pi_a(x)\nabla^i_x[\varphi_a(x), \tfrac{1}{2}\Pi_b(y)\Pi_b(y)] \\
&= i[\nabla^j\varphi_a(y)\nabla^j_y + \partial V(\varphi)/\partial\varphi_a]\delta^3(\mathbf{x}-\mathbf{y})\nabla^i_x\varphi_a(x) \\
&\quad -i\Pi_a(x)\Pi_a(y)\nabla^i_x\delta^3(\mathbf{x}-\mathbf{y})\ ,
\end{aligned} \tag{22.43}$$

$$\begin{aligned}
[T^{0i}(x), T^{0j}(y)] &= [\Pi_a(x)\nabla^i\varphi_a(x), \Pi_b(y)]\nabla^j\varphi_b(y) \\
&\quad +\Pi_b(y)\nabla^j_y[\Pi_a(x)\nabla^i\varphi_a(x), \varphi_b(y)] \\
&= i\Pi_a(x)\nabla^i_x\delta^3(\mathbf{x}-\mathbf{y})\nabla^j\varphi_a(y) \\
&\quad -i\Pi_a(y)\nabla^j_y\delta^3(\mathbf{x}-\mathbf{y})\nabla^i\varphi_a(x)\ .
\end{aligned} \tag{22.44}$$

b) If we integrate over d^3x and d^3y, we generate $[H, H]$, $[P^i, H]$, and $[P^i, P^j]$, respectively. The first of these vanishes (as it obviously must) because the $x \leftrightarrow y$ term cancels the first term after x and y become dummy integration variables. After some integrations by parts, it is easy to see $[P^i, P^j]$ vanishes as well. The hardest is $[P^i, H]$; we have

$$\begin{aligned}
[P^i, H] &= i\int d^3x\,[-\nabla^2\varphi_a\nabla^i\varphi_a + (\partial V/\partial\varphi_a)\nabla^i\varphi_a + \Pi_a\nabla^i\Pi_a] \\
&= i\int d^3x\,[\nabla^j\varphi_a\nabla_i\nabla^j\varphi_a + (\partial V/\partial\varphi_a)\nabla^i\varphi_a + \Pi_a\nabla^i\Pi_a] \\
&= i\int d^3x\,\nabla^i[\tfrac{1}{2}\nabla^j\varphi_a\nabla^j\varphi_a + V(\varphi) + \tfrac{1}{2}\Pi_a\Pi_a] \\
&= i\int d^3x\,\nabla^i T^{00}\ .
\end{aligned} \tag{22.45}$$

This vanishes (assuming suitable boundary conditions at spatial infinity) because it is the integral of a total derivative. This illustrates a useful general rule: since (as is easily checked) $[P^i, \varphi_a(x)] = i\nabla^i \varphi_a(x)$ and $[P^i, \Pi_a(x)] = i\nabla^i \Pi_a(x)$, any local function F of $\varphi_a(x)$ and $\Pi_a(x)$ and their spatial derivatives will obey

$$[P^i, F(x)] = i\nabla^i F(x) \; . \tag{22.46}$$

Next let us define

$$C^i \equiv \int d^3x \; x^i T^{00} \; , \tag{22.47}$$

$$D^{ij} \equiv \int d^3x \; x^i T^{0j} \; , \tag{22.48}$$

so that

$$K^i = C^i - x^0 P^i \; , \tag{22.49}$$

$$J^i = \varepsilon^{ijk} D^{jk} \; . \tag{22.50}$$

Using eqs. (22.46) and (22.47), we have

$$\begin{aligned} [P^i, C^j] &= i \int d^3x \; x^j \nabla^i T^{00} \\ &= -i \int d^3x \; (\nabla^i x^j) T^{00} \\ &= -i \int d^3x \; \delta^{ij} T^{00} \\ &= -i\delta^{ij} H \end{aligned} \tag{22.51}$$

Using eq. (22.49) and $[P^i, P^j] = 0$, we get $[P^i, K^j] = -i\delta^{ij} H$.

Now using eqs. (22.46) and (22.48), we have

$$\begin{aligned} [P^i, D^{jk}] &= i \int d^3x \; x^j \nabla^i T^{0k} \\ &= -i \int d^3x \; (\nabla^i x^j) T^{0k} \\ &= -i \int d^3x \; \delta^{ij} T^{0k} \\ &= -i\delta^{ij} P^k \; . \end{aligned} \tag{22.52}$$

Contracting with ε^{ljk} and using eq. (22.50), we get $[P^i, J^l] = -i\varepsilon^{lik} P^k$.

To see that $[J^l, H] = 0$, we multiply eq. (22.43) by x^j and integrate over d^3x and d^3y to get $[D^{ji}, H]$. Without the factor of x^j, the result would be $[P^i, H]$, which vanishes; thus a nonvanishing term can only result from an integration by parts that puts a ∇^i_x on x^j. Such a term would be proportional to δ^{ij}, and so vanishes when we contract with ε^{lji} to construct J^l.

To compute $[H, K^i] = [H, C^i]$, we multiply eq. (22.42) by y^i and integrate over d^3x and d^3y. Without the factor of y^i, the result would be $[H, H]$, which vanishes; thus a nonvanishing term can only result from an integration by parts that puts a ∇_y^i on y^i. The relevant terms yield

$$\begin{aligned}
[H, C^i] &= -\tfrac{1}{2}i \int d^3x\, d^3y\; y^i [\Pi_a(x)\nabla^j\varphi_a(y) + \nabla^j\varphi_a(y)\Pi_a(x)]\nabla_y^j \delta^3(\mathbf{x}-\mathbf{y}) \\
&= +\tfrac{1}{2}i \int d^3x\, d^3y\; (\nabla_y^j y^i)[\Pi_a(x)\nabla^j\varphi_a(y) + \nabla^j\varphi_a(y)\Pi_a(x)]\delta^3(\mathbf{x}-\mathbf{y}) \\
&= +\tfrac{1}{2}i \int d^3x\; \delta^{ij}[\Pi_a(x)\nabla^j\varphi_a(x) + \nabla^j\varphi_a(x)\Pi_a(x)] \\
&= +\tfrac{1}{2}i \int d^3x\; \delta^{ij}[2\Pi_a(x)\nabla^j\varphi_a(x) + \nabla^j\delta^3(\mathbf{0})] \\
&= -i\delta^{ij}P^j\;. \qquad (22.53)
\end{aligned}$$

Finally, to compute $[J^m, J^n] = \varepsilon^{mki}\varepsilon^{nlj}[D^{ki}, D^{lj}]$, we multiply eq. (22.44) by $x^k y^l$ and integrate over d^3x and d^3y. We get

$$\begin{aligned}
[D^{ki}, D^{lj}] &= i\int d^3x\, d^3y\; x^k y^l \Big[\Pi_a(x)\nabla_x^i\delta^3(\mathbf{x}-\mathbf{y})\nabla_y^j\varphi_a(y) - \Pi_a(y)\nabla_y^j\delta^3(\mathbf{x}-\mathbf{y})\nabla_x^i\varphi_a(x)\Big] \\
&= -i\int d^3x\, d^3y\; \Big[y^l\nabla_x^i(x^k\Pi_a(x))\nabla_y^j\varphi_a(y) - x^k\nabla_y^j(y^l\Pi_a(y))\nabla_x^i\varphi_a(x)\Big]\delta^3(\mathbf{x}-\mathbf{y}) \\
&= -i\int d^3x\; \Big[x^l\nabla^i(x^k\Pi_a)\nabla^j\varphi_a - x^k\nabla^j(x^l\Pi_a)\nabla^i\varphi_a\Big] \\
&= -i\int d^3x\; \Big[x^l(\delta^{ik}\Pi_a + x^k\nabla^i\Pi_a)\nabla^j\varphi_a - x^k(\delta^{jl}\Pi_a + x^l\nabla^j\Pi_a)\nabla^i\varphi_a\Big]\;. \qquad (22.54)
\end{aligned}$$

The δ^{ik} and δ^{jl} terms will vanish when we contract with $\varepsilon^{mki}\varepsilon^{nlj}$; thus we have

$$\begin{aligned}
[J^m, J^n] &= -i\varepsilon^{mki}\varepsilon^{nlj}\int d^3x\; \Big[x^k x^l\nabla^i\Pi_a\nabla^j\varphi_a - x^k x^l\nabla^j\Pi_a\nabla^i\varphi_a\Big] \\
&= +i\varepsilon^{mki}\varepsilon^{nlj}\int d^3x\; \Pi_a\Big[\nabla^i(x^k x^l\nabla^j\varphi_a) - \nabla^j(x^k x^l\nabla^i\varphi_a)\Big] \\
&= +i\varepsilon^{mki}\varepsilon^{nlj}\int d^3x\; \Pi_a\Big[(\delta^{ik}x^l + \delta^{il}x^k + x^k x^l\nabla^i)\nabla^j - (i\leftrightarrow j)\Big]\varphi_a \\
&= +i\varepsilon^{mki}\varepsilon^{nlj}\int d^3x\; \Pi_a(\delta^{il}x^k\nabla^i - \delta^{jk}x^l\nabla^i)\varphi_a \\
&= -i\varepsilon^{mki}\varepsilon^{nlj}\int d^3x\; (\delta^{il}x^k T^{0j} - \delta^{jk}x^l T^{0i}) \\
&= -i\varepsilon^{mki}\varepsilon^{nlj}(\delta^{il}D^{kj} - \delta^{jk}D^{li}) \\
&= -i\Big[(\varepsilon^{mki}\varepsilon^{nij})D^{kj} - (\varepsilon^{mji}\varepsilon^{nlj})D^{li}\Big] \\
&= -i\Big[(\delta^{jm}\delta^{kn} - \delta^{jk}\delta^{mn})D^{kj} - (\delta^{in}\delta^{lm} - \delta^{il}\delta^{mn})D^{li}\Big] \\
&= -i\Big[D^{nm} - D^{mn}\Big] \\
&= +i\varepsilon^{mnp}J^p\;. \qquad (22.55)
\end{aligned}$$

That was a lot of tedious work, but it's always good to confirm general arguments with specific calculations.

23 Discrete Symmetries: P, T, C, and Z

24 Nonabelian Symmetries

24.1) $R^T R = 1 \Rightarrow R_{ij}R_{ik} = \delta_{jk} \Rightarrow (\delta_{ij}+\theta_{ij})(\delta_{ik}+\theta_{ik}) = \delta_{jk} \Rightarrow \delta_{ij}\delta_{ik} + \delta_{ij}\theta_{ik} + \theta_{ij}\delta_{ik} + O(\theta^2) = \delta_{jk} \Rightarrow \delta_{jk} + \theta_{jk} + \theta_{kj} = \delta_{jk} \Rightarrow \theta_{jk} + \theta_{kj} = 0$.

24.2) Let $R = 1+\theta$, $R' = 1+\theta'$; then, keeping terms up to $O(\theta^2)$, we have

$$\begin{aligned} R'^{-1}R^{-1}R'R &= (1-\theta'+\theta'^2)(1-\theta+\theta^2)(1+\theta')(1+\theta) \\ &= 1 + \theta'\theta - \theta\theta' \,. \end{aligned} \tag{24.17}$$

This must be an orthogonal matrix of the form $1+\theta''$. Using eq. (24.6), we find $\theta^a\theta'^b[T^a,T^b] = i\theta''^c T^c$. Since the real parameters θ^a and θ^b can be chosen arbitrarily, this can only be true if $[T^a,T^b] = if^{abc}T^c$, where the coefficients f^{abc} are real.

24.3) a) From eq. (22.6), we have

$$\begin{aligned} \theta^a j^{\mu a} &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}\delta\varphi_i \\ &= (-\partial^\mu\varphi_i)(-i\theta^a(T^a)_{ij}\varphi_j) \,, \end{aligned} \tag{24.18}$$

which yields

$$j^{a\mu} = i\partial^\mu\varphi_i(T^a)_{ij}\varphi_j \,. \tag{24.19}$$

b) $Q = \int d^3y\; j^{0a}(y)$, and $j^{0a} = i\partial^0\varphi_j(T^a)_{jk}\varphi_k = -i\dot\varphi_j(T^a)_{jk}\varphi_k = -i\Pi_j(T^a)_{jk}\varphi_k$. Q is time independent, so we can take $y^0 = x^0$. Then $[\varphi_i(x), j^{0a}(y)] = -i[\varphi_i(x),\Pi_j(y)](T^a)_{jk}\varphi_k(y) = \delta^3(\mathbf{x}-\mathbf{y})\delta_{ij}(T^a)_{jk}\varphi_k(y)$. Integrating over d^3y yields $[\varphi_i(x),Q^a] = (T^a)_{ik}\varphi_k(x)$.

c) Consider $[[\varphi_i,Q^a],Q^b] = (T^a)_{ij}[\varphi_j,Q^b] = (T^a)_{ij}(T^b)_{jk}\varphi_k = (T^aT^b)_{ik}\varphi_k$; swapping a and b yields $[[\varphi_i,Q^b],Q^a] = (T^bT^a)_{ik}\varphi_k$. Subtracting, we get

$$[[\varphi_i,Q^a],Q^b] - [[\varphi_i,Q^b],Q^a] = [T^a,T^b]_{ik}\varphi_k = if^{abc}(T^c)_{ik}\varphi_k \,. \tag{24.20}$$

The left-hand side of eq. (24.20) can be rewritten as $[[\varphi_i,Q^a],Q^b] + [[Q^b,\varphi_i],Q^a]$, and by the Jacobi identity, this equals $-[[Q^a,Q^b],\varphi_i]$, which can be rewritten as $[\varphi_i,[Q^a,Q^b]]$. Thus we have

$$[\varphi_i,[Q^a,Q^b]] = if^{abc}(T^c)_{ik}\varphi_k \,. \tag{24.21}$$

Contracting $[\varphi_i,Q^c] = (T^c)_{ik}\varphi_k$ with if^{abc} also yields the right-hand side of eq. (24.21). Thus, $[Q^a,Q^b] - if^{abc}Q^c$ commutes with $\varphi_a(x)$, and (since Q^a is time independent) also with $\dot\varphi_a(x) = \Pi_a(x)$. Therefore $[Q^a,Q^b] - if^{abc}Q^c$ must equal a constant tensor with two antisymmetric adjoint indices. There is no such invariant symbol for SO(N), and so the constant tensor must vanish.

24.4) Let $S = 1+\theta$; then $S\eta S^T = \eta$ implies $\theta\eta + \eta\theta^T = 0$. Let us write

$$\theta = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{24.22}$$

Then

$$\theta\eta + \eta\theta^T = \begin{pmatrix} -B+B^T & A+D^T \\ -A-D^T & C-C^T \end{pmatrix}, \tag{24.23}$$

and this vanishes only if $B^T = B$, $C^T = C$, and $D^T = A$. Thus we can choose the N^2 elements of A freely, while B and C each have $\frac{1}{2}N(N+1)$ independent elements. Thus the total number of independent matrix elements of θ is $2N^2 + N = \frac{1}{2}(2N)(2N+1)$, so this is the number of generators of $\mathrm{Sp}(2N)$.

25 Unstable Particles and Resonances

26 Infrared Divergences

27 Other Renormalization Schemes

27.1) For notational convenience, let $t = \ln \mu$; Then we have $d\alpha/dt = b_1 \alpha^2$, or equivalently $dt = d\alpha/b_1\alpha^2$. We also have $dm/dt = c_1 \alpha m$, or equivalently $dm/m = c_1 \alpha \, dt = (c_1/b_1) d\alpha/\alpha$. Integrating, we find $\ln(m_2/m_1) = (c_1/b_1) \ln(\alpha_2/\alpha_1)$, which implies eq. (27.29).

28 The Renormalization Group

28.1) From section 31, we have

$$Z_\varphi = 1 + O(\lambda^2) \, , \tag{28.49}$$

$$Z_m = 1 + \frac{\lambda}{16\pi^2}\frac{1}{\varepsilon} + O(\lambda^2) \, , \tag{28.50}$$

$$Z_\lambda = 1 + \frac{3\lambda}{16\pi^2}\frac{1}{\varepsilon} + O(\lambda^2) \, , \tag{28.51}$$

and $\beta(\lambda) = 3\lambda^2/16\pi^2 + O(\lambda^3)$. Following section 28, we define $M_n(\lambda)$ via

$$\sum_{n=1}^{\infty} \frac{M_n(\lambda)}{\varepsilon^n} = \ln(Z_m^{1/2} Z_\varphi^{-1/2}) \, . \tag{28.52}$$

Then we have $\gamma_m(\lambda) = \lambda M_1'(\lambda)$. Using eqs. (28.49) and (28.50), we find $M_1(\lambda) = \lambda/32\pi^2$, and so $\gamma_m(\lambda) = \lambda/32\pi^2$. Similarly, we define $Z_\varphi = 1 + a_1(\lambda)/\varepsilon + \ldots$, and then $\gamma(\lambda) = -\frac{1}{2}\lambda a_1'(\lambda)$; we have $a_1(\lambda) = O(\lambda^2)$, so $\gamma(\lambda) = O(\lambda^2)$.

28.2) From problems 14.6 and 16.2, we have

$$Z_\varphi = 1 + O(\lambda^2) \, , \tag{28.53}$$

$$Z_m = 1 + \frac{\lambda}{8\pi^2}\frac{1}{\varepsilon} + O(\lambda^2) \, , \tag{28.54}$$

$$Z_\lambda = 1 + \frac{5\lambda}{16\pi^2}\frac{1}{\varepsilon} + O(\lambda^2) \, . \tag{28.55}$$

These yield $\beta(\lambda) = 5\lambda^2/16\pi^2 + O(\lambda^3)$, $\gamma_m(\lambda) = \lambda/16\pi^2 + O(\lambda^2)$, and $\gamma(\lambda) = O(\lambda^2)$.

28.3) No term linear in χ is needed because it enters the lagrangian only in even powers, and so we cannot draw a diagram with just one external χ line. Thus its VEV is automatically zero. Equivalently, the lagrangian is invariant under the Z_2 symmetry $\chi \to -\chi$, and the argument at the end of section 23 applies.

a) Use a solid line for χ and a dashed line for φ. The one-loop and counterterm diagrams contributing to the χ propagator are

Note that the symmetry factor of the loop diagram is $S = 1$. Following the analysis in section 14, the corresponding contributions to the self-energy are

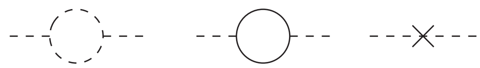
$$\Pi_\chi(k^2) = -\frac{h^2}{(4\pi)^3}\left(\frac{2}{\varepsilon} + \ldots\right)\int_0^1 dx\, D\,[1 + O(\varepsilon)] - (Z_\chi - 1)k^2 - (Z_M - 1)M^2 \, , \tag{28.56}$$

where $D = x(1-x)k^2 + xm^2 + (1-x)M^2$. We have $\int_0^1 dx\ D = \frac{1}{6}k^2 + \frac{1}{2}m^2 + \frac{1}{2}M^2$, and so cancelation of the $1/\varepsilon$ terms requires

$$Z_\chi = 1 - \frac{h^2}{3(4\pi)^3}\frac{1}{\varepsilon} \, , \tag{28.57}$$

$$Z_M = 1 - \frac{h^2}{(4\pi)^3}\left(1 + \frac{m^2}{M^2}\right)\frac{1}{\varepsilon} \, , \tag{28.58}$$

at the one-loop level. For the φ propagator, we have



Each loop diagram has a symmetry factor of $S = 2$. Thus we have

$$\Pi_\varphi(k^2) = -\frac{1}{2}\frac{1}{(4\pi)^3}\left(\frac{2}{\varepsilon}+\ldots\right)\Big[g^2(\tfrac{1}{6}k^2+m^2)+h^2(\tfrac{1}{6}k^2+M^2)+\ldots\Big]-(Z_\varphi{-}1)k^2-(Z_m{-}1)m^2 \tag{28.59}$$

and so cancelation of the $1/\varepsilon$ terms requires

$$Z_\varphi = 1-\frac{1}{6(4\pi)^3}\Big(g^2+h^2\Big)\frac{1}{\varepsilon}\,, \tag{28.60}$$

$$Z_M = 1-\frac{1}{(4\pi)^3}\left(g^2+\frac{M^2}{m^2}h^2\right)\frac{1}{\varepsilon}\,. \tag{28.61}$$

at the one-loop level. For the φ^3 vertex, the contributing one-loop diagrams are

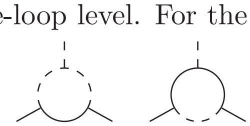
Using our results from section 16, we have

$$\mathbf{V}_{\varphi^3} = Z_g g+\frac{1}{(4\pi)^3}\left(\frac{1}{\varepsilon}+\ldots\right)\Big(g^3+h^3\Big)\,, \tag{28.62}$$

and so cancelation of the $1/\varepsilon$ terms requires

$$Z_g = 1-\frac{1}{(4\pi)^3}\left(g^2+\frac{h^3}{g}\right)\frac{1}{\varepsilon} \tag{28.63}$$

at the one-loop level. For the $\varphi\chi^2$ vertex, the contributing one-loop diagrams are



Using our results from section 16, we have

$$\mathbf{V}_{\varphi\chi^2} = Z_h h+\frac{1}{(4\pi)^3}\left(\frac{1}{\varepsilon}+\ldots\right)\Big(gh^2+h^3\Big)\,, \tag{28.64}$$

and so cancelation of the $1/\varepsilon$ terms requires

$$Z_h = 1-\frac{1}{(4\pi)^3}\Big(gh+h^2\Big)\frac{1}{\varepsilon} \tag{28.65}$$

at the one-loop level.

b) We have

$$\ln g_0 = G+\ln g+\tfrac{1}{2}\varepsilon\ln\tilde{\mu}\,, \tag{28.66}$$

$$\ln h_0 = H+\ln h+\tfrac{1}{2}\varepsilon\ln\tilde{\mu}\,, \tag{28.67}$$

where $G = \sum_n G_n/\varepsilon^n = \ln(Z_\varphi^{-3/2} Z_g)$ and $H = \sum_n H_n/\varepsilon^n = \ln(Z_\varphi^{-1/2} Z_\chi^{-1} Z_h)$, and

$$\begin{aligned} G_1 &= \frac{1}{(4\pi)^3}\Big[-\tfrac{3}{2}(-\tfrac{1}{6}g^2 - \tfrac{1}{6}h^2) + (-g^2 - h^3/g)\Big] \\ &= \frac{1}{(4\pi)^3}\Big[-\tfrac{3}{4}g^2 + \tfrac{1}{4}h^2 - h^3/g\Big] , \end{aligned} \tag{28.68}$$

$$\begin{aligned} H_1 &= \frac{1}{(4\pi)^3}\Big[-\tfrac{1}{2}(-\tfrac{1}{6}g^2 - \tfrac{1}{6}h^2) - (-\tfrac{1}{3}h^2) + (-gh - h^2)\Big] \\ &= \frac{1}{(4\pi)^3}\Big[\tfrac{1}{12}g^2 - gh - \tfrac{7}{12}h^2\Big] . \end{aligned} \tag{28.69}$$

Differentiating eq. (28.66) with respect to μ, multiplying by $g\mu$, and denoting $\mu\, d/d\mu$ with a dot, we find

$$\begin{aligned} 0 &= g\dot{G} + \dot{g} + \tfrac{1}{2}\varepsilon g \\ &= g\frac{\partial G}{\partial g}\dot{g} + g\frac{\partial G}{\partial h}\dot{h} + \dot{g} + \tfrac{1}{2}\varepsilon g \\ &= \left(1 + g\,\frac{\partial G}{\partial g}\right)\dot{g} + g\frac{\partial G}{\partial h}\dot{h} + \tfrac{1}{2}\varepsilon g . \end{aligned} \tag{28.70}$$

Similarly differentiating eq. (28.67), we find

$$0 = \left(1 + h\,\frac{\partial H}{\partial h}\right)\dot{h} + h\frac{\partial H}{\partial g}\dot{g} + \tfrac{1}{2}\varepsilon h . \tag{28.71}$$

Eqs. (28.70) and (28.71) can be combined into

$$0 = \begin{pmatrix} 1 + g\frac{\partial G}{\partial g} & g\frac{\partial G}{\partial h} \\ h\frac{\partial H}{\partial g} & 1 + h\frac{\partial H}{\partial h} \end{pmatrix} \begin{pmatrix} \dot{g} \\ \dot{h} \end{pmatrix} + \tfrac{1}{2}\varepsilon \begin{pmatrix} g \\ h \end{pmatrix} . \tag{28.72}$$

Solving for $\dot{g}$ and $\dot{h}$, we have

$$\begin{pmatrix} \dot{g} \\ \dot{h} \end{pmatrix} = -\tfrac{1}{2}\varepsilon \begin{pmatrix} 1 + g\frac{\partial G}{\partial g} & g\frac{\partial G}{\partial h} \\ h\frac{\partial H}{\partial g} & 1 + h\frac{\partial H}{\partial h} \end{pmatrix}^{-1} \begin{pmatrix} g \\ h \end{pmatrix} . \tag{28.73}$$

Formally expanding in powers of $1/\varepsilon$, we find

$$\begin{pmatrix} \dot{g} \\ \dot{h} \end{pmatrix} = -\tfrac{1}{2}\varepsilon \begin{pmatrix} g \\ h \end{pmatrix} + \tfrac{1}{2} \begin{pmatrix} g\frac{\partial G_1}{\partial g} & g\frac{\partial G_1}{\partial h} \\ h\frac{\partial H_1}{\partial g} & h\frac{\partial H_1}{\partial h} \end{pmatrix} \begin{pmatrix} g \\ h \end{pmatrix} + O(\varepsilon^{-1}) . \tag{28.74}$$

The $O(\varepsilon^{-1})$ and higher terms must vanish in a renormalizable theory.

c) Plugging eqs. (28.68) and (28.69) into eq. (28.74), we find

$$\beta_g(g,h) = \frac{1}{(4\pi)^3}\Big[-\tfrac{3}{4}g^3 + \tfrac{1}{4}gh^2 - h^3\Big] , \tag{28.75}$$

$$\beta_h(g,h) = \frac{1}{(4\pi)^3}\Big[-\tfrac{7}{12}h^3 - gh^2 + \tfrac{1}{12}hg^2\Big] . \tag{28.76}$$

d) If we change the sign of g, we can compensate by changing the sign of both φ and h. But once the sign of g is fixed, we cannot compensate for changing the sign of h. We see in the formula for β_g that the sign of h is relevant.

If β_g/g and β_h/h are both negative, the theory is asymptotically free: both couplings get weaker at high energy. Let us define $r \equiv h/g$; then we have $\beta_g/g = (g^2/4(4\pi)^3)(-3+r-4r^3)$, which is negative for $r > -1$, and $\beta_h/h = (g^2/12(4\pi)^3)(-7r^2 - 12r + 1)$, which is negative for $r < -\frac{1}{7}(\sqrt{43}+6) = -1.78$ and for $r > \frac{1}{7}(\sqrt{43}-6) = 0.08$. Thus the theory is asymptotically free for $r > 0.08$.

29 Effective Field Theory

29.1) a) Let a dot denote $d/d\ln\Lambda$, so that $\dot{g} = b_1 g^2 + b_2 g^3 + \ldots$. Inverting $\tilde{g} = g + c_2 g^2 + \ldots$, we have $g = \tilde{g} - c_2\tilde{g}^2 + \ldots$ We now have

$$\begin{aligned} \dot{g} &= \dot{\tilde{g}} - 2c_2\tilde{g}\dot{\tilde{g}} + \ldots \\ &= (1 - 2c_2\tilde{g} + \ldots)\dot{\tilde{g}}\ , \end{aligned} \tag{29.44}$$

or, rearranging,

$$\begin{aligned} \dot{\tilde{g}} &= (1 + 2c_2\tilde{g} + \ldots)^{-1}\dot{g} \\ &= (1 + 2c_2\tilde{g} + \ldots)(b_1 g^2 + b_2 g^3 + \ldots) \\ &= (1 + 2c_2\tilde{g} + \ldots)[b_1(\tilde{g} - c_2\tilde{g}^2)^2 + b_2(\tilde{g} - c_2\tilde{g}^2)^3 + \ldots] \\ &= b_1\tilde{g}^2 + b_2\tilde{g}^3 + \ldots\ . \end{aligned} \tag{29.45}$$

b) Just make everything into a matrix: we then have $\dot{g}_i = b_{2,ijk}g_j g_k + b_{3,ijkl}g_j g_k g_l + \ldots$ and $g_i = \tilde{g}_i - c_{2,ijk}\tilde{g}_j\tilde{g}_k + \ldots$. Everything in part (a) still goes through.

29.2) a) We use the relation $\tilde{\mathbf{\Delta}}(k^2)^{-1} = \Delta(k^2)^{-1} - \Pi(k^2)$, and only fields with $\Lambda < |\ell| < \Lambda_0$ circulate in the loop in fig. 14.1; in this case, $\tilde{\mathbf{\Delta}}(k^2)$ is the propagator with a cutoff Λ, and differentiating $\tilde{\mathbf{\Delta}}(k^2)^{-1}$ with respect to k^2 (and setting $k^2 = 0$) yields $Z(\Lambda)$, the coefficient of the kinetic term when the cutoff is Λ. The vertex factor is $-Z^{3/2}(\Lambda_0)g(\Lambda_0)$, and the tree-level propagator is $\Delta(k^2) = 1/[Z(\Lambda_0)k^2]$. We thus have $Z(\Lambda) = Z(\Lambda_0) - \Pi'(k^2)$. At the one-loop level,

$$\Pi(k^2) = \tfrac{1}{2}[Z^{3/2}(\Lambda_0)g(\Lambda_0)]^2 \int_\Lambda^{\Lambda_0} \frac{d^6\ell}{(2\pi)^6}\,\frac{1}{[Z(\Lambda_0)\ell^2][Z(\Lambda_0)(\ell+k)^2]}\ , \tag{29.46}$$

where the one-half is a symmetry factor. Differentiating with respect to k^2, setting $k^2 = 0$, and plugging into $Z(\Lambda) = Z(\Lambda_0) - \Pi'(k^2)$ then yields the first unnumbered equation in the problem text.

Using Feynman's formula, we have $1/(\ell^2(\ell+k)^2) = \int_0^1 dx\,(q^2 + x(1-x)k^2)^{-2}$, where $q = \ell + xk$. Differentiating with respect to k^2 and setting k to zero yields $-2\int_0^1 dx\,x(1-x)(\ell^2)^{-3} = -\frac{1}{3}\ell^{-6}$. Thus we have

$$\begin{aligned} \Pi'(k^2) &= -\tfrac{1}{2}Z(\Lambda_0)g^2(\Lambda_0)\,\frac{\Omega_3}{3(2\pi)^6}\int_{\Lambda_0}^{\Lambda}\frac{\ell^5\,d\ell}{\ell^6} \\ &= -Z(\Lambda_0)\,\frac{g^2(\Lambda_0)}{6(4\pi)^3}\,\ln(\Lambda_0/\Lambda)\ , \end{aligned} \tag{29.47}$$

where $\Omega_d = 2\pi^{d/2}/\Gamma(\frac{1}{2}d)$, so that $\Omega_6 = \pi^3$. Plugging this into $Z(\Lambda) = Z(\Lambda_0) - \Pi'(k^2)$ then yields

$$Z(\Lambda) = Z(\Lambda_0)\left(1 + \frac{g^2(\Lambda_0)}{6(4\pi)^3}\,\ln(\Lambda_0/\Lambda)\right). \tag{29.48}$$

The vertex correction works similarly; with a cutoff Λ, we have $\mathbf{V}_3(0,0,0) = -Z^{3/2}(\Lambda)g(\Lambda)$, and only fields with $\Lambda < |\ell| < \Lambda_0$ circulate in the loop in fig. 16.1. This yields the second unnumbered equation in the problem text, which evaluates to

$$g(\Lambda) = \frac{Z^{3/2}(\Lambda_0)}{Z^{3/2}(\Lambda)}\, g(\Lambda_0)\left(1 + \frac{g^2(\Lambda_0)}{(4\pi)^3}\,\ln(\Lambda_0/\Lambda)\right). \tag{29.49}$$

b) Using eq. (29.48) for $Z(\Lambda)$, and expanding in powers of $g(\Lambda_0)$, we find

$$\begin{aligned} g(\Lambda) &= g(\Lambda_0)\left(1 + \Big((-\tfrac{3}{2})(\tfrac{1}{6}) + 1\Big)\frac{g^2(\Lambda_0)}{(4\pi)^3}\,\ln(\Lambda_0/\Lambda)\right) \\ &= g(\Lambda_0)\left(1 + \frac{3}{4}\,\frac{g^2(\Lambda_0)}{(4\pi)^3}\,\ln(\Lambda_0/\Lambda)\right). \end{aligned} \tag{29.50}$$

Differentiating with respect to $\ln\Lambda$ and then setting $\Lambda_0 = \Lambda$, we find

$$\frac{d}{d\ln\Lambda}\, g(\Lambda) = -\frac{3}{4}\,\frac{g^3(\Lambda)}{(4\pi)^3}\,. \tag{29.51}$$

Multiplying by $2g/(4\pi)^3$, the left-hand side becomes $d\alpha/d\ln\Lambda$, where $\alpha = g^2/(4\pi)^3$, and the right-hand side becomes $-\frac{3}{2}\alpha^2$. This agrees with our result in section 28.

30 Spontaneous Symmetry Breaking

31 Broken Symmetry and Loop Corrections

32 Spontaneous Breaking of Continuous Symmetries

32.1) a) From eq. (22.14), $j^\mu = i\partial^\mu\varphi^\dagger\varphi - i\partial^\mu\varphi\varphi^\dagger$, so that $j^0 = -i\dot\varphi^\dagger\varphi + i\dot\varphi\varphi^\dagger = -i\Pi\varphi - i\Pi^\dagger\varphi^\dagger$, where $[\varphi(x),\Pi(y)] = [\varphi^\dagger(x),\Pi^\dagger(y)] = i\delta^3(\mathbf{x}-\mathbf{y})$ at equal times, and all other commutators vanish. Thus $[\varphi(x), j^0(y)] = -i[\varphi(x),\Pi(y)]\varphi(y) = \delta^3(\mathbf{x}-\mathbf{y})\varphi(y)$, and integrating over d^3y yields $[\varphi(x),Q] = \varphi(x)$. Next, let $F(\alpha) \equiv e^{-i\alpha Q}\varphi e^{+i\alpha Q}$, and note that $F'(\alpha) = e^{-i\alpha Q} i[\varphi,Q]e^{+i\alpha Q}$. Since $[\varphi,Q] = \varphi$, this becomes $F'(\alpha) = ie^{-i\alpha Q}\varphi e^{+i\alpha Q} = iF(\alpha)$. Therefore $F^{(n)}(\alpha) = i^n F(\alpha)$, and $F^{(n)}(0) = i^n F(0) = i^n\varphi$. Thus, by Taylor expansion, $F(\alpha) = \sum_{n=0}^\infty F^{(n)}(0)\alpha^n/n! = \varphi\sum_{n=0}^\infty i^n\alpha^n/n! = \varphi e^{+i\alpha}$.

b) Since $[H,Q] = 0$, $He^{-i\alpha Q}|\theta\rangle = 0$, so $e^{-i\alpha Q}|\theta\rangle$ must be a linear combination of vacua. Then, since $e^{+i\alpha Q}\varphi e^{-i\alpha Q} = e^{-i\alpha}\varphi$, we have $\langle\theta|e^{+i\alpha Q}\varphi e^{-i\alpha Q}|\theta\rangle = e^{-i\alpha}\langle\theta|\varphi|\theta\rangle$; using eq. (32.5), this becomes $\langle\theta|e^{+i\alpha Q}\varphi e^{-i\alpha Q}|\theta\rangle = \frac{1}{\sqrt2}ve^{-i(\theta+\alpha)} = \langle\theta{+}\alpha|\varphi|\theta{+}\alpha\rangle$.

c) Expanding in powers of α, we get $(1 - i\alpha Q)|\theta\rangle = |\theta\rangle + \alpha(d/d\theta)|\theta\rangle$; the second term on the right-hand side is not zero.

32.2) If $Q^a|0\rangle = 0$, then $\langle 0|Q^a = 0$, and $\langle 0|[\varphi_i, Q^a]\rangle = \langle 0|\varphi_i Q^a|0\rangle - \langle 0|Q^a\varphi_i|0\rangle = 0$. Thus if $\langle 0|[\varphi_i,Q^a]|0\rangle = (T^a)_{ij}\langle 0|\varphi_j|0\rangle = \frac{1}{\sqrt2}(T^a)_{ij}v_j \neq 0$, then $Q^a|0\rangle \neq 0$.

32.3) a) $j^\mu = -i\partial^\mu\varphi\varphi^\dagger + \text{h.c.}$; plugging in eq. (32.8), we get $j^\mu = -v(1+\rho/v)^2\partial^\mu\chi$. In free-field theory, we then find $\langle k|j^\mu(x)|0\rangle = ivk^\mu e^{-ikx}$, so $f = v$.

b) We can compute corrections by treating $j^\mu(x)$ as a vertex, and drawing Feynman diagrams with one external χ line. The single tree diagram just attaches that line to the vertex with vertex factor ivk^μ, yielding $f = v$. Loop corrections will modify this to $f = v(1 + O(\lambda))$.

33 Representations of the Lorentz Group

33.1) $A^{\mu\nu} = \frac{1}{2}(B^{\mu\nu} - B^{\nu\mu})$, $T = g_{\mu\nu}B^{\mu\nu}$, $S^{\mu\nu} = \frac{1}{2}(B^{\mu\nu} + B^{\nu\mu}) - \frac{1}{4}g^{\mu\nu}T$.

33.2) $4[N_i, N_j] = [J_i, J_j] - i[J_i, K_j] - i[K_i, J_j] - [K_i, K_j] = i\varepsilon_{ijk}(J_k - iK_k - iK_k + J_k) = 4i\varepsilon_{ijk}N_k$.

$4[N_i^\dagger, N_j^\dagger] = [J_i, J_j] + i[J_i, K_j] + i[K_i, J_j] - [K_i, K_j] = i\varepsilon_{ijk}(J_k + iK_k + iK_k + J_k) = 4i\varepsilon_{ijk}N_k^\dagger$.

$4[N_i, N_j^\dagger] = [J_i, J_j] + i[J_i, K_j] - i[K_i, J_j] + [K_i, K_j] = i\varepsilon_{ijk}(J_k + iK_k - iK_k - J_k) = 0$.

34 Left- and Right-Handed Spinor Fields

34.1) Without the spin indices, this is the same as problem 2.8; we have $U(\Lambda)^{-1}\psi_a(x)U(\Lambda) = \psi_a(x) + \frac{i}{2}\delta\omega_{\mu\nu}[\psi_a(x), M^{\mu\nu}]$ on the LHS, and $L_a{}^b(\Lambda)\psi_b(\Lambda^{-1}x) = [\delta_a{}^b + \frac{i}{2}\delta\omega_{\mu\nu}(S_{\rm L}^{\mu\nu})_a{}^b][\psi_b(x) + \frac{i}{2}\delta\omega_{\mu\nu}\mathcal{L}^{\mu\nu}\psi_b(x)] = \psi_a(x) + \frac{i}{2}\delta\omega_{\mu\nu}[\delta_a{}^b\mathcal{L}^{\mu\nu} + (S_{\rm L}^{\mu\nu})_a{}^b]\psi_b(x)$ on the RHS; matching coefficients of $\delta\omega_{\mu\nu}$ yields eq. (34.6).

34.2) The commutation relations of the S's are the same as those of the M's. In problem 2.4, we showed that the commutation relations of the M's are equivalent to eqs. (33.11–13). Let us define $\mathcal{J}^i \equiv \frac{1}{2}\varepsilon^{ijk}S_{\rm L}^{jk}$; from eq. (34.9) we see that $\mathcal{J}^k = \frac{1}{2}\sigma^k$. Let us also define $\mathcal{K}^k \equiv S_{\rm L}^{k0}$; from eq. (34.10) we see that $\mathcal{K}^k = \frac{1}{2}i\sigma^k$. Eqs. (33.11–13) for $\mathcal{J}$ and $\mathcal{K}$ then follow immediately from the Pauli-matrix commutation relations, $[\sigma^i, \sigma^j] = 2i\varepsilon^{ijk}\sigma^k$.

34.3) Consider $\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\alpha\beta\gamma\sigma}$; the indices on each Levi-Civita symbol must be all different to get a nonzero result. Consider (for example) $\varepsilon^{123\sigma}\varepsilon_{\alpha\beta\gamma\sigma}$; then only $\sigma = 0$ contributes to the sum; and then the second symbol is nonzero only if $\alpha\beta\gamma$ is a permutation of 123. If it is an even permutation, then the result is $\varepsilon^{1230}\varepsilon_{1230} = (-1)(+1) = -1$, and if it is an odd permutation, then the result is $+1$. More generally, the result is -1 if $\alpha\beta\gamma$ is an even permutation of $\mu\nu\rho$, and $+1$ if $\alpha\beta\gamma$ is an odd permutation of $\mu\nu\rho$. This is equivalent to eq. (34.44).

To get eq. (34.45), we contract with $\delta^\gamma{}_\rho$ and use $\delta^\gamma{}_\rho\delta^\rho{}_\gamma = 4$. To get eq. (34.46), we further contract with $\delta^\beta{}_\nu$.

34.4) Consider first a tensor with N totally symmetric undotted indices, and no dotted indices. Because the indices are totally symmetric, we can put them in a standard order, with all 1's before all 2's. Each independent component is then labeled by an integer $k = 0, \ldots, N$ that specifies the number of 1's, and so the number of independent components is $N{+}1$.

Next, using

$$[C_{ab\ldots c}(0), N_3] = \tfrac{1}{2}(\sigma_3)_a{}^d C_{db\ldots c}(0) + \tfrac{1}{2}(\sigma_3)_b{}^d C_{ad\ldots c}(0) + \ldots + \tfrac{1}{2}(\sigma_3)_c{}^d C_{ab\ldots d}(0)\;, \qquad (34.47)$$

which follows from eqs. (34.7), (34.9), $J^i = N^i + N^{\dagger i}$, and $[C_{ab\ldots c}(x), N^{\dagger i}] = 0$, we have

$$[C_{11\ldots 2}(0), N_3] = \Big(\tfrac{1}{2}k - \tfrac{1}{2}(N{-}k)\Big)C_{11\ldots 2}(0)\;. \qquad (34.48)$$

We see that the allowed values of N_3 are $-\frac{1}{2}N, -\frac{1}{2}N{+}1, \ldots, +\frac{1}{2}N$, corresponding to $k = 0, 1, \ldots, N$. Thus these $N{+}1$ components correspond to a single irreducible representation with dimension $2n{+}1 = N{+}1$. If we now add M completely symmertic dotted indices, these are treated independently, and form a single irreducible representation with dimension $2n'{+}1 = M{+}1$. Thus, overall the representation is $(N{+}1, M{+}1)$.

35 Manipulating Spinor Indices

35.1) $\bar{\sigma}^{\mu\dot{a}a} = \varepsilon^{ac}\varepsilon^{\dot{a}\dot{c}}\sigma^{\mu}_{c\dot{c}} = -\varepsilon^{ac}\sigma^{\mu}_{c\dot{c}}\varepsilon^{\dot{c}\dot{a}} = -[(i\sigma_2)(\sigma^\mu)(i\sigma_2)]^{a\dot{a}} = [(\sigma_2\sigma^\mu\sigma_2)^{\rm T}]^{\dot{a}a}$. Then for $\mu = 3$ we have $\sigma_2\sigma^3\sigma_2 = -(\sigma_2)^2\sigma^3 = -\sigma^3$, and $(\sigma^3)^{\rm T} = \sigma^3$; the same is true for $\mu = 1$. For $\mu = 2$, we have $\sigma_2\sigma^2\sigma_2 = (\sigma_2)^2\sigma^2 = +\sigma^2$, and $(\sigma^2)^{\rm T} = -\sigma^2$. For $\mu = 0$, we have $\sigma_2 I\sigma_2 = (\sigma_2)^2 = I$, and $I^{\rm T} = I$. Thus we have $\bar{\sigma}^{0\dot{a}a} = I$ and $\bar{\sigma}^{i\dot{a}a} = -\sigma^i$.

35.2) $(S_{\rm L}^{\mu\nu})_a{}^b = \frac{i}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)_a{}^b$. Suppressing spin indices, we have $S_{\rm L}^{12} = \frac{i}{4}(\sigma^1\bar{\sigma}^2 - \sigma^2\bar{\sigma}^1) = \frac{i}{4}((\sigma^1)(-\sigma^2) - (\sigma^2)(-\sigma^1)) = -\frac{i}{4}[\sigma^1,\sigma^2] = \frac{1}{2}\sigma_3$, and cyclic permutations. Also, $S_{\rm L}^{k0} = \frac{i}{4}(\sigma^k\bar{\sigma}^0 - \sigma^0\bar{\sigma}^k) = \frac{i}{4}((\sigma^k)(I) - (I)(-\sigma^k)) = \frac{i}{2}\sigma^k$.

35.3) $(S_{\rm R}^{\mu\nu})^{\dot{a}}{}_{\dot{c}} = -\frac{i}{4}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)^{\dot{a}}{}_{\dot{c}}$. Suppressing spin indices, we have $S_{\rm R}^{12} = -\frac{i}{4}(\bar{\sigma}^1\sigma^2 - \bar{\sigma}^2\sigma^1) = -\frac{i}{4}((-\sigma^1)(\sigma^2) - (-\sigma^2)(-\sigma^1)) = \frac{i}{4}[\sigma^1,\sigma^2] = -\frac{1}{2}\sigma_3$, and cyclic permutations. Also, $S_{\rm R}^{k0} = -\frac{i}{4}(\bar{\sigma}^k\sigma^0 - \bar{\sigma}^0\sigma^k) = -\frac{i}{4}((-\sigma^k)(I) - (I)(\sigma^k)) = \frac{i}{2}\sigma^k$. Eq. (34.17) can be written as $(S_{\rm R}^{\mu\nu})^{\dot{a}}{}_{\dot{c}} = -[(S_{\rm L}^{\mu\nu})^a{}_b]^* = -[(S_{\rm L}^{\mu\nu})_b{}^a]^{{\rm T}*} = -[(S_{\rm L}^{\mu\nu})_b{}^a]^\dagger$. Comparing with our results in the previous problem, and using the hermiticity of the Pauli matrices, we see that $(S_{\rm R}^{\mu\nu})^{\dot{a}}{}_{\dot{c}} = -[(S_{\rm L}^{\mu\nu})_b{}^a]^\dagger$ is satisfied.

35.4) $\varepsilon^{ac}\varepsilon^{\dot{a}\dot{c}}\sigma^\mu_{a\dot{a}}\sigma^\nu_{c\dot{c}} = \sigma^\mu_{a\dot{a}}\bar{\sigma}^{\nu\dot{a}a} = {\rm Tr}(\sigma^\mu\bar{\sigma}^\nu)$. If $\mu = 0$ and $\nu = 0$, we get ${\rm Tr}(I) = 2 = -2g^{00}$. If $\mu = 0$ and $\nu = i$ or vice versa, we get the trace of σ^i, which vanishes. If $\mu = i$ and $\nu = j$, we get $-{\rm Tr}(\sigma^i\sigma^j) = -2\delta^{ij} = -2g^{ij}$.

36 Lagrangians for Spinor Fields

36.1) In problem 2.9, we showed that eqs. (36.56–57) hold for the vector representation. The result, however, must be representation independent.

36.2) $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = i\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}$
$= \begin{pmatrix} i\sigma_1\sigma_2\sigma_3 & 0 \\ 0 & -i\sigma_1\sigma_2\sigma_3 \end{pmatrix} = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}.$

36.3) a) We have $(\chi_1^\dagger\bar\sigma^\mu\chi_2)(\chi_3^\dagger\bar\sigma_\mu\chi_4) = \bar\sigma^{\mu\dot a a}\bar\sigma_\mu^{\dot c c}\,\chi^\dagger_{1\dot a}\chi_{2a}\chi^\dagger_{3\dot c}\chi_{4c}$. Then we use $\bar\sigma^{\mu\dot a a}\bar\sigma_\mu^{\dot c c} = -2\varepsilon^{ac}\varepsilon^{\dot a\dot c}$ and $\chi^\dagger_{1\dot a}\chi_{2a}\chi^\dagger_{3\dot c}\chi_{4c} = -\chi^\dagger_{1\dot a}\chi^\dagger_{3\dot c}\chi_{2a}\chi_{4c}$ along with $\varepsilon^{ac}\chi_c = \chi^a$ and its dotted counterpart to get $(\chi_1^\dagger\bar\sigma^\mu\chi_2)(\chi_3^\dagger\bar\sigma_\mu\chi_4) = 2\chi^\dagger_{1\dot a}\chi_3^{\dagger\dot a}\chi_{2a}\chi_4^a = -2\chi^\dagger_{1\dot a}\chi_3^{\dagger\dot a}\chi_2^a\chi_{4a} = -2(\chi_1^\dagger\chi_3^\dagger)(\chi_2\chi_4)$, which is eq. (36.58). Then we use $\chi_2\chi_4 = \chi_4\chi_2$, and go backwards throught these steps to get the right-hand side of eq. (36.59).

b) Using eqs. (36.7), (36.22), (36.45), and (36.60), we find $\overline\Psi_1\gamma^\mu P_{\rm L}\Psi_2 = \chi_1^\dagger\bar\sigma^\mu\chi_2$, $\overline\Psi_1 P_{\rm R}\Psi_3^{\rm C} = \chi_1^\dagger\chi_3^\dagger$, and $\overline\Psi{}_4^{\rm C} P_{\rm L}\Psi_2 = \chi_4\chi_2$, which yield eqs. (36.61–62) from eqs. (36.58–59).

c) In terms of Weyl fields, we have $\overline\Psi_1\gamma^\mu P_{\rm R}\Psi_2 = \xi_1\sigma^\mu\xi_2^\dagger = -\xi_2^\dagger\bar\sigma^\mu\xi_1 = -\overline\Psi{}_2^{\rm C}\gamma^\mu P_{\rm L}\Psi_1^{\rm C}$, $\overline\Psi_1 P_{\rm L}\Psi_2 = \xi_1\chi_2 = \chi_2\xi_1 = \overline\Psi{}_2^{\rm C}P_{\rm L}\Psi_1^{\rm C}$, and $\overline\Psi_1 P_{\rm R}\Psi_2 = \chi_1^\dagger\xi_2^\dagger = \xi_2^\dagger\chi_1^\dagger = \overline\Psi{}_2^{\rm C}P_{\rm R}\Psi_1^{\rm C}$.

36.4) a) This form for $T^{\mu\nu}$ is identical to eq. (22.29). The derivation is unchanged if the index a is replaced with the Lorentz index A.

b) For $\Lambda = 1+\delta\omega$, the Lorentz transformation $\varphi_A(x) \to L_A{}^B(\Lambda)\varphi_B(\Lambda^{-1}x)$ becomes $\varphi_A(x) \to (\delta_A{}^B + \frac{i}{2}\delta\omega_{\nu\rho}(S^{\nu\rho})_A{}^B)(\varphi_B(x) - \delta\omega_{\nu\rho}x^\rho\partial^\nu\varphi_B(x))$, so that $\delta\varphi_A = \delta\omega_{\nu\rho}(-x^\rho\partial^\nu\varphi_A + \frac{i}{2}(S^{\nu\rho})_A{}^B)\varphi_B$. Also, $\mathcal{L}(x) \to \mathcal{L}(\Lambda^{-1}x)$ implies $\delta\mathcal{L} = -\delta\omega_\nu{}^\rho x_\rho\partial^\nu\mathcal{L} = \partial^\mu(-\delta\omega_{\nu\rho}g^{\mu\nu}x^\rho\mathcal{L})$; we then identify $K^\mu = -\delta\omega_{\nu\rho}g^{\mu\nu}x^\rho\mathcal{L}$. Using eq. (22.27), we then have

$$\begin{aligned} j^\mu &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_A)}\,\delta\varphi_A - K^\mu \\ &= \delta\omega_{\nu\rho}\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_A)}\,(-x^\rho\partial^\nu\varphi_A) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_A)}\,\tfrac{i}{2}(S^{\nu\rho})_A{}^B\varphi_B + g^{\mu\nu}x^\rho\mathcal{L}\right] \\ &= \delta\omega_{\nu\rho}\left[x^\rho T^{\mu\nu} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_A)}\,\tfrac{i}{2}(S^{\nu\rho})_A{}^B\varphi_B\right] \\ &= -\tfrac{1}{2}\delta\omega_{\nu\rho}\left[x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu} - i\,\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_A)}(S^{\nu\rho})_A{}^B\varphi_B\right], \end{aligned} \tag{36.81}$$

and we identify the object in square brackets as

$$\mathcal{M}^{\mu\nu\rho} \equiv x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu} + B^{\mu\nu\rho}\ , \tag{36.82}$$

where

$$B^{\mu\nu\rho} \equiv -i\,\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_A)}(S^{\nu\rho})_A{}^B\varphi_B\ . \tag{36.83}$$

c) Consider $\partial_\mu \mathcal{M}^{\mu\nu\rho}$; we have $\partial_\mu(x^\nu T^{\mu\rho}) = \delta_\mu{}^\nu T^{\mu\rho} + x^\nu \partial_\mu T^{\mu\nu} = T^{\nu\rho} + 0 = T^{\nu\rho}$, and so $0 = \partial_\mu \mathcal{M}^{\mu\nu\rho} = T^{\nu\rho} - T^{\rho\nu} + \partial_\mu B^{\mu\nu\rho}$.

d) We have $\Theta^{\mu\nu} \equiv T^{\mu\nu} + \frac{1}{2}\partial_\rho(B^{\rho\mu\nu} - B^{\mu\rho\nu} - B^{\nu\rho\mu})$. Note that because (by definition) $S^{\mu\nu} = -S^{\nu\mu}$, eq. (36.83) implies $B^{\rho\mu\nu} = -B^{\rho\nu\mu}$. Note also that the last two terms in $\Theta^{\mu\nu}$ are symmetric on $\mu \leftrightarrow \nu$. Thus we have $\Theta^{\mu\nu} - \Theta^{\nu\mu} = T^{\mu\nu} - T^{\nu\mu} + \partial_\rho B^{\rho\mu\nu}$, which vanishes according to the result of part (c).

Next consider $\partial_\mu \Theta^{\mu\nu} = \partial_\mu T^{\mu\nu} + \frac{1}{2}\partial_\mu\partial_\rho(B^{\rho\mu\nu} - B^{\mu\rho\nu} - B^{\nu\rho\mu}) = \frac{1}{2}\partial_\mu\partial_\rho(B^{\rho\mu\nu} - B^{\mu\rho\nu} - B^{\nu\rho\mu})$. Note that $B^{\rho\mu\nu} - B^{\mu\rho\nu} + B^{\nu\rho\mu}$ is antisymmetric on $\mu \leftrightarrow \rho$, and therefore vanishes when acted on by the symmetric derivative combination $\partial_\mu\partial_\rho$.

$\Theta^{0\nu} = T^{0\nu} + \frac{1}{2}\partial_\rho(B^{\rho 0\nu} - B^{0\rho\nu} - B^{\nu\rho 0}) = T^{0\nu} + \frac{1}{2}\partial_i(B^{i0\nu} - B^{0i\nu} - B^{\nu i0})$. The integral over d^3x of $\frac{1}{2}\partial_i(\ldots)$ vanishes (assuming suitable boundary conditions at spatial infinity) because it is a total divergence. Therefore $P^\nu = \int d^3x\, T^{0\nu} = \int d^3x\, \Theta^{0\nu}$.

e) Recall from part (c) that $\partial_\mu(x^\nu \Theta^{\mu\rho}) = \Theta^{\nu\rho}$ if $\partial_\mu \Theta^{\mu\nu} = 0$. We have $\Xi^{\mu\nu\rho} \equiv x^\nu \Theta^{\mu\rho} - x^\rho \Theta^{\mu\nu}$, and so $\partial_\mu \Xi^{\mu\nu\rho} = \Theta^{\nu\rho} - \Theta^{\rho\nu} = 0$.

$$\begin{aligned}
\Xi^{\mu\nu\rho} &= x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu} + \tfrac{1}{2}x^\nu \partial_\sigma(B^{\sigma\mu\rho} - B^{\mu\sigma\rho} - B^{\rho\sigma\mu}) - \tfrac{1}{2}x^\rho \partial_\sigma(B^{\sigma\mu\nu} - B^{\mu\sigma\nu} - B^{\nu\sigma\mu}) \\
&= \mathcal{M}^{\mu\nu\rho} - B^{\mu\nu\rho} + \tfrac{1}{2}x^\nu \partial_\sigma(B^{\sigma\mu\rho} - B^{\mu\sigma\rho} - B^{\rho\sigma\mu}) - \tfrac{1}{2}x^\rho \partial_\sigma(B^{\sigma\mu\nu} - B^{\mu\sigma\nu} - B^{\nu\sigma\mu})\ ,
\end{aligned}$$

and so

$$\Xi^{0\nu\rho} = \mathcal{M}^{0\nu\rho} - B^{0\nu\rho} + \tfrac{1}{2}x^\nu \partial_i(B^{i0\rho} - B^{0i\rho} - B^{\rho i0}) + \tfrac{1}{2}x^\rho \partial_i(B^{i0\nu} - B^{0i\nu} - B^{\nu i0})\ .$$

Now using $x^\nu \partial_i(\ldots) = \partial_i[x^\nu(\ldots)] - (\ldots)\partial_i x^\nu = \partial_i[x^\nu(\ldots)] - (\ldots)\delta_i{}^\nu$, we get

$$\begin{aligned}
\Xi^{0\nu\rho} &= \mathcal{M}^{0\nu\rho} - B^{0\nu\rho} - \tfrac{1}{2}(B^{\nu 0\rho} - B^{0\nu\rho} - B^{\rho\nu 0}) + \tfrac{1}{2}(B^{\rho 0\nu} - B^{0\rho\nu} - B^{\nu\rho 0}) + \partial_i[\ldots] \\
&= \mathcal{M}^{0\nu\rho} - \tfrac{1}{2}(B^{0\nu\rho} + B^{0\rho\nu}) - \tfrac{1}{2}(B^{\nu 0\rho} + B^{\nu\rho 0}) + \tfrac{1}{2}(B^{\rho 0\nu} + B^{\rho\nu 0}) + \partial_i[\ldots] \\
&= \mathcal{M}^{0\nu\rho} + \partial_i[\ldots]\ .
\end{aligned}$$

Since the last term is a total divergence, $M^{\nu\rho} = \int d^3x\, \mathcal{M}^{0\nu\rho} = \int d^3x\, \Xi^{0\nu\rho}$.

36.5) a) The transformation matrix must be orthogonal to preserve the mass term, hence the symmetry is O(N).

b) A Majorana field is equivalent to a Weyl field, hence the symmetry is U(N).

c) Combining the results of parts (a) and (b), the symmetry is O(N).

d) A Dirac field is equivalent to two Weyl fields, hence the symmetry is U($2N$).

e) Combining the results of parts (a) and (d), the symmetry is O($2N$).

37 Canonical Quantization of Spinor Fields I

37.1) Eq. (37.13) follows immediately because all components of χ anticommute with all components of $\xi^\dagger$. To get eq. (37.14), we write

$$\begin{aligned} \{\Psi_\alpha, \Psi_\beta\} &= \begin{pmatrix} \{\chi_c, \xi^a\} & \{\chi_c, \chi^\dagger_{\dot a}\} \\ \{\xi^{\dagger\dot c}, \xi^a\} & \{\xi^{\dagger\dot c}, \chi^\dagger_{\dot a}\} \end{pmatrix} \\ &= \begin{pmatrix} 0 & \sigma^0_{c\dot a} \\ \bar\sigma^{0\dot c a} & 0 \end{pmatrix} \delta^3(\mathbf{x}-\mathbf{y}) \\ &= \gamma^0 \delta^3(\mathbf{x}-\mathbf{y}) \; . \end{aligned} \tag{37.32}$$

where we used eqs. (37.7) and (37.8) to get the second line.

38 Spinor Technology

38.1) In a basis where A is diagonal (with diagonal entries ± 1), $\exp(cA) = (\cosh c) + (\sinh c)A$ is obvious. We have $2iK^j = -\gamma^j\gamma^0 = \begin{pmatrix} -\sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix}$, which obviously has eigenvalues ± 1, and so

$$\exp(i\eta\hat{\mathbf{p}}\cdot\mathbf{K}) = \begin{pmatrix} (\cosh\frac{1}{2}\eta) - (\sinh\frac{1}{2}\eta)\hat{\mathbf{p}}\cdot\boldsymbol{\sigma} & 0 \\ 0 & (\cosh\frac{1}{2}\eta) + (\sinh\frac{1}{2}\eta)\hat{\mathbf{p}}\cdot\boldsymbol{\sigma} \end{pmatrix}, \tag{38.42}$$

where $\sinh\eta = |\mathbf{p}|/m$, $\cosh\eta = E/m$, $\cosh\frac{1}{2}\eta = \sqrt{(E+m)/2m}$, $\sinh\frac{1}{2}\eta = \sqrt{(E-m)/2m}$. Also, using the usual angles θ and ϕ to specify $\hat{\mathbf{p}}$, we have $\hat{\mathbf{p}}\cdot\boldsymbol{\sigma} = \begin{pmatrix} \cos\theta & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & -\cos\theta \end{pmatrix}$. We can then act on $u_\pm(\mathbf{0})$ and $v_\pm(\mathbf{0})$ as given by eq. (38.6) to get $u_\pm(\mathbf{p})$ and $v_\pm(\mathbf{p})$

38.2) From the explicit form of the gamma matrices, we can see that $(\gamma^0)^\dagger = \gamma^0$ and $(\gamma^j)^\dagger = -\gamma^j$. Also, since $\beta = \gamma^0$ numerically, the gamma matrix anticommutation relations imply $\beta^2 = 1$, $\beta\gamma^0 = \gamma^0\beta$, and $\beta\gamma^j = -\gamma^j\beta$. Therefore $\overline{\gamma^0} = \beta(\gamma^0)^\dagger\beta = \beta\gamma^0\beta = \gamma^0\beta^2 = \gamma^0$, and $\overline{\gamma^j} = \beta(\gamma^j)^\dagger\beta = -\beta\gamma^j\beta = \gamma^j\beta^2 = \gamma^j$. Thus, $\overline{\gamma^\mu} = \gamma^\mu$.

From the explicit form of γ_5, we see that it is hermitian. We also have $\{\gamma^\mu, \gamma_5\} = 0$, since γ^μ commutes with one gamma matrix in $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and anticommutes with the other three. Therefore $\overline{\gamma_5} = \beta(\gamma_5)^\dagger\beta = \beta\gamma_5\beta = -\gamma_5\beta^2 = -\gamma_5$. Since $\bar{i} = -i$, we have $\overline{i\gamma_5} = i\gamma_5$.

The remaining formulae in eq. (38.15) can be found from these by using $\overline{AB} = \beta(AB)^\dagger\beta = \beta B^\dagger A^\dagger\beta = \beta B^\dagger\beta\beta A^\dagger\beta = \overline{B}\,\overline{A}$.

38.3) Subtract eq. (38.19) from eq. (38.20), sandwich between $\overline{u}_s(\mathbf{p})$ and $v_{s'}(\mathbf{p}')$, and use $\not{p}'v_{s'}(\mathbf{p}') = mv_s(\mathbf{p}')$ and $\overline{u}_s(\mathbf{p})\not{p} = -m\overline{u}_s(\mathbf{p})$ to get

$$2m\,\overline{u}_s(\mathbf{p})\gamma^\mu v_{s'}(\mathbf{p}') = \overline{u}_s(\mathbf{p})\Big[(p-p')^\mu + 2iS^{\mu\nu}(p+p')_\nu\Big]v_s(\mathbf{p}')\,. \tag{38.43}$$

If we now set $\mathbf{p}' = -\mathbf{p}$ (which implies $p'^0 = p^0$) and $\mu = 0$, and remember that $S^{00} = 0$, all terms on the right-hand side vanish. This yields the first equation in (38.22). An identical derivation applies to the second; or, bar-conjugate the first and relabel.

38.4) Add eqs. (38.19) and (38.20), multiply on the right by γ_5, sandwich between $\overline{u}_{s'}(\mathbf{p}')$ and $u_s(\mathbf{p})$ or $\overline{v}_{s'}(\mathbf{p}')$ and $v_s(\mathbf{p})$, and use eq. (38.16). Then use $\not{p}\gamma_5 = -\gamma_5\not{p}$, which follows from the definition of γ_5, followed by eq. (38.1).

39 Canonical Quantization of Spinor Fields II

39.1) Substituting in the mode expansions for Ψ and $\overline{\Psi}$, we have

$$Q = \sum_{s,s'} \int \widetilde{dp}\,\widetilde{dp}'\,d^3x \left(b^\dagger_{s'}(\mathbf{p}')\overline{u}_{s'}(\mathbf{p}')e^{-ip'x} + d_{s'}(\mathbf{p}')\overline{v}_{s'}(\mathbf{p}')e^{ip'x}\right) \times \left(b_s(\mathbf{p})\gamma^0 u_s(\mathbf{p})e^{ipx} + d^\dagger_s(\mathbf{p})\gamma^0 v_s(\mathbf{p})e^{-ipx}\right). \tag{39.44}$$

Note that this is the same as the first equality in eq. (39.21) for H, except that (1) a factor of ω is missing and (2) the $d^\dagger_s(\mathbf{p})$ term has a plus sign rather than a minus sign. Thus we conclude that the final formula for Q is also the same, with these changes, and so

$$\begin{aligned} Q &= \sum_{s=\pm} \int \widetilde{dp}\left[b^\dagger_s(\mathbf{p})b_s(\mathbf{p}) + d_s(\mathbf{p})d^\dagger_s(\mathbf{p})\right] \\ &= \sum_{s=\pm} \int \widetilde{dp}\left[b^\dagger_s(\mathbf{p})b_s(\mathbf{p}) - d^\dagger_s(\mathbf{p})d_s(\mathbf{p})\right] + \text{constant}\,. \end{aligned} \tag{39.45}$$

39.2) We have

$$\begin{aligned} J_z b^\dagger_s(\mathbf{p})|0\rangle &= \int d^3x\, e^{ipx}\, J_z\overline{\Psi}(x)\gamma^0 u_s(\mathbf{p})|0\rangle \\ &= \int d^3x\, e^{ipx}\,[J_z,\overline{\Psi}(x)]\gamma^0 u_s(\mathbf{p})|0\rangle \\ &= \int d^3x\, e^{ipx}\,[M^{12},\overline{\Psi}(x)]\gamma^0 u_s(\mathbf{p})|0\rangle\,, \end{aligned} \tag{39.46}$$

where we used $J_z|0\rangle = 0$ in the second line. Barring $[\Psi, M^{\mu\nu}] = -i(x^\mu\partial^\nu - x^\nu\partial^\mu)\Psi + S^{\mu\nu}\Psi$ yields $[M^{\mu\nu},\overline{\Psi}] = i(x^\mu\partial^\nu - x^\nu\partial^\mu)\overline{\Psi} + \overline{\Psi}S^{\mu\nu}$, and so

$$J_z b^\dagger_s(\mathbf{p})|0\rangle = \int d^3x\, e^{ipx}\Big[i(x^1\partial^2 - x^2\partial^1)\overline{\Psi}(x) + \overline{\Psi}(x)S^{12}\Big]\gamma^0 u_s(\mathbf{p})|0\rangle\,. \tag{39.47}$$

For $\mathbf{p} = p\hat{\mathbf{z}}$, we can integrate by parts in the first term and get zero (more precisely, a surface term that we assume vanishes via suitable boundary conditions at spatial infinity). In the second term, we use $S^{12}\gamma^0 = \gamma^0 S^{12}$, and $u_s(\mathbf{p}) = \exp(i\eta K^3)u_s(\mathbf{0})$ with $K^3 = \frac{i}{2}\gamma^3\gamma^0$. We have $[S^{12},K^3] = 0$, and so $S^{12}u_s(\mathbf{p}) = \exp(i\eta K^3)S^{12}u_s(\mathbf{0}) = +\frac{1}{2}s\exp(i\eta K^3)u_s(\mathbf{0}) = +\frac{1}{2}s\,u_s(\mathbf{p})$. This leaves

$$\begin{aligned} J_z b^\dagger_s(\mathbf{p})|0\rangle &= +\tfrac{1}{2}s\int d^3x\, e^{ipx}\,\overline{\Psi}(x)\gamma^0 u_s(\mathbf{p})|0\rangle \\ &= +\tfrac{1}{2}s\, b^\dagger_s(\mathbf{p})|0\rangle\,. \end{aligned} \tag{39.48}$$

Similarly,

$$\begin{aligned} J_z d^\dagger_s(\mathbf{p})|0\rangle &= \int d^3x\, e^{ipx}\,\overline{v}_s(\mathbf{p})\gamma^0 J_z\Psi(x)|0\rangle \\ &= \int d^3x\, e^{ipx}\,\overline{v}_s(\mathbf{p})\gamma^0[J_z,\Psi(x)]|0\rangle \\ &= \int d^3x\, e^{ipx}\,\overline{v}_s(\mathbf{p})\gamma^0\Big[i(x^1\partial^2 - x^2\partial^1)\Psi(x) - S^{12}\Psi(x)\Big]|0\rangle \\ &= -\int d^3x\, e^{ipx}\,\overline{v}_s(\mathbf{p})\gamma^0 S^{12}\Psi(x)|0\rangle \end{aligned} \tag{39.49}$$

We have $\overline{v}_s(\mathbf{p})\gamma^0 S^{12} = \overline{S^{12}\gamma^0 v_s(\mathbf{p})}$ and $S^{12}\gamma^0 v_s(\mathbf{p}) = \gamma^0 \exp(i\eta K^3) S^{12} v_s(\mathbf{0}) = -\frac{1}{2}s\gamma^0 v_s(\mathbf{p})$. Therefore

$$\begin{aligned} J_z d_s^\dagger(\mathbf{p})|0\rangle &= \tfrac{1}{2}s \int d^3x\, e^{ipx}\, \overline{v}_s(\mathbf{p})\gamma^0 \Psi(x)|0\rangle \\ &= \tfrac{1}{2}s\, d_s^\dagger(\mathbf{p})|0\rangle\ . \end{aligned} \tag{39.50}$$

39.3) In problem 3.3, we showed that $U(\Lambda)^{-1}a(\mathbf{k})U(\Lambda) = a(\Lambda^{-1}\mathbf{k})$ for a scalar, where $a(\mathbf{k})$ is the coefficient of e^{ikx} in the mode expansion. Since $\sum_s b_s(\mathbf{p})u_s(\mathbf{p})$ is the coefficient of e^{ipx} in the mode expansion of a Dirac or Majorana field, we similarly have

$$U(\Lambda)^{-1}{\textstyle\sum_s} b_s(\mathbf{p})u_s(\mathbf{p})U(\Lambda) = D(\Lambda){\textstyle\sum_s} b_s(\Lambda^{-1}\mathbf{p})u_s(\Lambda^{-1}\mathbf{p})\ , \tag{39.51}$$

where the matrix $D(\Lambda)$ comes from the transformation rule for $\Psi(x)$; $D(\Lambda)$ acts on $u_s(\Lambda^{-1}\mathbf{p})$, and we have $D(\Lambda)u_s(\Lambda^{-1}\mathbf{p}) = u_s(\mathbf{p})$. Then, we multiply on the left with $\overline{u}_{s'}(\mathbf{p})$ and use $\overline{u}_{s'}(\mathbf{p})u_s(\mathbf{p}) = 2m\delta_{s's}$ to get $U(\Lambda)^{-1}b_s(\mathbf{p})U(\Lambda) = b_s(\Lambda^{-1}\mathbf{p})$; the first equation in (39.39) then follows by hermitian conjugation. A similar analysis applies to the coefficient of e^{-ipx}, yielding the second equation in (39.39). Eq. (39.40) then follows as in problem 3.3.

39.4) a) Using eq. (39.39), we have

$$\begin{aligned} U(\Lambda)^{-1}\Psi^+(x)U(\Lambda) &= \int \widetilde{dp}\, U(\Lambda)^{-1}b_s(\mathbf{p})U(\Lambda)u_s(\mathbf{p})e^{ipx} \\ &= \int \widetilde{dp}\, b_s(\Lambda^{-1}\mathbf{p})u_s(\mathbf{p})e^{ipx} \\ &= \int \widetilde{dp}\, b_s(\mathbf{p})u_s(\Lambda\mathbf{p})e^{i(\Lambda p)x} \\ &= \int \widetilde{dp}\, b_s(\mathbf{p})D(\Lambda)u_s(\mathbf{p})e^{ip(\Lambda^{-1}x)} \\ &= D(\Lambda)\Psi^+(\Lambda^{-1}x)\ , \end{aligned} \tag{39.52}$$

where the third equality follows from changing the integration variable from p to Λp, and using the invariance of $\widetilde{dp}$. The fourth equality uses $(\Lambda p)x = \Lambda^\mu{}_\nu p^\nu x_\mu = p^\nu(\Lambda^{-1})_\nu{}^\mu x_\mu = p(\Lambda^{-1}x)$. The same steps work for $\Psi^-(x)$.

b) $[\Psi^+(x)]^\dagger = \overline{\Psi^+(x)}\beta = \sum_s \int \widetilde{dp}\, b_s^\dagger(\mathbf{p})\overline{u}_s(\mathbf{p})\beta e^{-ipx} = \sum_s \int \widetilde{dp}\, b_s^\dagger(\mathbf{p})v_s^{\mathrm{T}}(\mathbf{p})\mathcal{C}\beta e^{-ipx} = [\Psi^-(x)]^{\mathrm{T}}\mathcal{C}\beta$.

c) We have

$$\begin{aligned} [\Psi_\alpha^+(x), \Psi_\beta^-(y)]_\mp &= \sum_{s,s'} \int \widetilde{dp}\,\widetilde{dp}'\, [b_s(\mathbf{p}), b_{s'}^\dagger(\mathbf{p}')]_\mp\, u_s(\mathbf{p})_\alpha v_{s'}(\mathbf{p}')_\beta\, e^{i(px-p'y)} \\ &= \sum_s \int \widetilde{dp}\ u_s(\mathbf{p})_\alpha v_s(\mathbf{p})_\beta\ e^{ip(x-y)}\ . \end{aligned} \tag{39.53}$$

Now we use $v_s^{\mathrm{T}}(\mathbf{p}) = -\overline{u}_s(\mathbf{p})\mathcal{C}$ to get

$$\begin{aligned} [\Psi_\alpha^+(x), \Psi_\beta^-(y)]_\mp &= -\int \widetilde{dp}\, {\textstyle\sum_s} [u_s(\mathbf{p})\overline{u}_s(\mathbf{p})\mathcal{C}]_{\alpha\beta}\, e^{ip(x-y)} \\ &= -\int \widetilde{dp}\ [(-\not{p} + m)\mathcal{C}]_{\alpha\beta}\, e^{ip(x-y)} \end{aligned}$$

$$
\begin{aligned}
&= -[(i\partial\!\!\!/_x + m)\mathcal{C}]_{\alpha\beta} \int \widetilde{dp}\; e^{ip(x-y)} \\
&= -[(i\partial\!\!\!/_x + m)\mathcal{C}]_{\alpha\beta}\, C(r)\;, \qquad (39.54)
\end{aligned}
$$

where $r^2 = (x-y)^2 > 0$ and $C(r) = mK_1(mr)/4\pi^2 r$; see section 4.

d) Swapping $x \leftrightarrow y$ and $\alpha \leftrightarrow \beta$ in the second line of eq. (39.54), we get

$$
[\Psi^+_\beta(y), \Psi^-_\alpha(x)]_\mp = -\int \widetilde{dp}\; [(-p\!\!\!/ + m)\mathcal{C}]_{\beta\alpha}\, e^{-ip(x-y)}\;. \qquad (39.55)
$$

Using $\mathcal{C}^{\rm T} = -\mathcal{C}$ and $(\gamma^\mu \mathcal{C})^{\rm T} = \mathcal{C}^{\rm T}\gamma^{\mu{\rm T}} = (-\mathcal{C})(-\mathcal{C}^{-1}\gamma^\mu\mathcal{C}) = \gamma^\mu\mathcal{C}$, we find that $[(-p\!\!\!/ + m)\mathcal{C}]_{\beta\alpha} = [(-p\!\!\!/ - m)\mathcal{C}]_{\alpha\beta}$. So

$$
\begin{aligned}
[\Psi^+_\beta(y), \Psi^-_\alpha(x)]_\mp &= -\int \widetilde{dp}\; [(-p\!\!\!/ - m)\mathcal{C}]_{\alpha\beta}\, e^{-ip(x-y)} \\
&= +[(i\partial\!\!\!/_x + m)\mathcal{C}]_{\alpha\beta} \int \widetilde{dp}\; e^{-ip(x-y)}\;. \qquad (39.56)
\end{aligned}
$$

Since $(x-y)^2 > 0$, we can work in a frame where $x^0 = y^0$; then $p(x-y) = \mathbf{p}\cdot(\mathbf{x}-\mathbf{y})$, and we can change $\mathbf{p}$ to $-\mathbf{p}$ in the integrand to get

$$
\begin{aligned}
[\Psi^+_\beta(y), \Psi^-_\alpha(x)]_\mp &= +[(i\partial\!\!\!/_x + m)\mathcal{C}]_{\alpha\beta} \int \widetilde{dp}\; e^{ip(x-y)} \\
&= +[(i\partial\!\!\!/_x + m)\mathcal{C}]_{\alpha\beta}\, C(r) \\
&= -[\Psi^+_\alpha(x), \Psi^-_\beta(y)]_\mp\;. \qquad (39.57)
\end{aligned}
$$

e) We note that $[\Psi^+_\alpha(x), \Psi^+_\beta(y)]_\mp$, $[\Psi^-_\alpha(x), \Psi^-_\beta(y)]_\mp$, $[\Psi^+_\alpha(x), \overline{\Psi}{}^-_\beta(y)]_\mp$, $[\Psi^-_\alpha(x), \overline{\Psi}{}^+_\beta(y)]_\mp$ vanish because $[b,b]_\mp$ and $[b^\dagger, b^\dagger]_\mp$ vanish. Therefore, with $\Psi(x) = \Psi^+(x) + \lambda\Psi^-(x)$, we have

$$
\begin{aligned}
[\Psi_\alpha(x), \Psi_\beta(y)]_\mp &= [\Psi^+_\alpha(x), \lambda\Psi^-_\beta(y)]_\mp + [\lambda\Psi^-_\alpha(x), \Psi^+_\beta(y)]_\mp \\
&= \lambda[\Psi^+_\alpha(x), \Psi^-_\beta(y)]_\mp \mp \lambda[\Psi^+_\beta(y), \Psi^-_\alpha(x)]_\mp \\
&= \lambda(1 \pm 1)[\Psi^+_\alpha(x), \Psi^-_\beta(y)]_\mp\;, \qquad (39.58)
\end{aligned}
$$

where we used $[A,B]_\mp = \mp[B,A]_\mp$ to get the second line, and eq. (39.57) to get the third. This can vanish if and only if we choose the lower sign, that is, if we use anticommutators. We also have

$$
\begin{aligned}
[\Psi^+_\alpha(x), \overline{\Psi}{}^+_\beta(y)]_\mp &= \sum_{s,s'} \int \widetilde{dp}\,\widetilde{dp}{}'\, [b_s(\mathbf{p}), b^\dagger_{s'}(\mathbf{p}')]_\mp\, u_s(\mathbf{p})_\alpha \overline{u}_{s'}(\mathbf{p}')_\beta\, e^{i(px-p'y)} \\
&= \int \widetilde{dp}\; \textstyle\sum_s u_s(\mathbf{p})_\alpha \overline{u}_s(\mathbf{p})_\beta\; e^{ip(x-y)} \\
&= \int \widetilde{dp}\; (-p\!\!\!/ + m)_{\alpha\beta}\; e^{ip(x-y)} \\
&= (i\partial\!\!\!/_x + m)_{\alpha\beta}\, C(r)\;, \qquad (39.59)
\end{aligned}
$$

and

$$
\begin{aligned}
[\Psi^-_\alpha(x),\overline{\Psi}^-_\beta(y)]_\mp &= \sum_{s,s'}\int \widetilde{dp}\,\widetilde{dp}'\,[b^\dagger_s(\mathbf{p}),b_{s'}(\mathbf{p}')]_\mp\, v_s(\mathbf{p})_\alpha \overline{v}_{s'}(\mathbf{p}')_\beta\, e^{-i(px-p'y)} \\
&= \mp\int \widetilde{dp}\,\textstyle\sum_s v_s(\mathbf{p})_\alpha \overline{v}_s(\mathbf{p})_\beta\, e^{-ip(x-y)} \\
&= \mp\int \widetilde{dp}\,(-\not{p}-m)_{\alpha\beta}\, e^{-ip(x-y)} \\
&= \pm(i\not{\partial}_x+m)_{\alpha\beta}\, C(r)\ .
\end{aligned} \tag{39.60}
$$

Therefore

$$
\begin{aligned}
[\Psi_\alpha(x),\overline{\Psi}_\beta(y)]_\mp &= [\Psi^+_\alpha(x),\overline{\Psi}^+_\beta(y)]_\mp + |\lambda|^2[\Psi^-_\alpha(x),\overline{\Psi}^-_\beta(y)]_\mp \\
&= (1\pm|\lambda|^2)(i\not{\partial}_x+m)_{\alpha\beta}\, C(r)\ ,
\end{aligned} \tag{39.61}
$$

which vanishes only for the lower sign (anticommutators) and $|\lambda| = 1$.

40 Parity, Time Reversal, and Charge Conjugation

40.1) We have $P^{-1}\overline{\Psi}A\Psi P = \overline{\Psi}\beta A\beta\Psi$, where $A = S^{\mu\nu} = \frac{i}{2}\gamma^\mu\gamma^\nu$ or $A = iS^{\mu\nu}\gamma_5 = -\frac{1}{2}\gamma^\mu\gamma^\nu\gamma_5$ (with $\mu \neq \nu$). Using $\beta\gamma^i = -\gamma^i\beta$, $\beta\gamma^0 = \gamma^0\beta$, and $\beta^2 = 1$, we have $\beta\gamma^i\gamma^j\beta = +\gamma^i\gamma^j$ and $\beta\gamma^0\gamma^i\beta = -\gamma^0\gamma^i$. Therefore

$$P^{-1}\overline{\Psi}S^{\mu\nu}\Psi P = +\mathcal{P}^\mu{}_\rho\mathcal{P}^\nu{}_\sigma\overline{\Psi}S^{\rho\sigma}\Psi \; . \tag{40.49}$$

Using $\beta\gamma_5 = -\gamma_5\beta$, we have $\beta\gamma^i\gamma^j\gamma_5\beta = -\gamma^i\gamma^j\gamma_5$ and $\beta\gamma^0\gamma^i\gamma_5\beta = +\gamma^0\gamma^i\gamma_5$. Therefore

$$P^{-1}\overline{\Psi}iS^{\mu\nu}\gamma_5\Psi P = -\mathcal{P}^\mu{}_\rho\mathcal{P}^\nu{}_\sigma\overline{\Psi}iS^{\rho\sigma}\gamma_5\Psi \; . \tag{40.50}$$

We have $T^{-1}\overline{\Psi}A\Psi T = \overline{\Psi}\widetilde{A}\Psi$, where we have defined $\widetilde{A} \equiv \gamma_5\mathcal{C}^{-1}A^*\mathcal{C}\gamma_5$. Note that $\widetilde{AB} = \widetilde{A}\widetilde{B}$. From eq. (40.40), we have $\widetilde{\gamma^\mu} = -\mathcal{T}^\mu{}_\rho\gamma^\rho$ and $\widetilde{i\gamma_5} = -i\gamma_5$. Therefore $\widetilde{S^{\mu\nu}} = -\frac{i}{2}\widetilde{\gamma^\mu}\widetilde{\gamma^\nu} = -\mathcal{T}^\mu{}_\rho\mathcal{T}^\nu{}_\sigma S^{\rho\sigma}$ and $\widetilde{S^{\mu\nu}i\gamma_5} = \widetilde{S^{\mu\nu}}\widetilde{i\gamma_5} = +\mathcal{T}^\mu{}_\rho\mathcal{T}^\nu{}_\sigma S^{\rho\sigma}i\gamma_5$, and so

$$\begin{aligned} T^{-1}\overline{\Psi}S^{\mu\nu}\Psi T &= -\mathcal{T}^\mu{}_\rho\mathcal{T}^\nu{}_\sigma\overline{\Psi}S^{\rho\sigma}\Psi \; , \\ T^{-1}\overline{\Psi}iS^{\mu\nu}\gamma_5\Psi T &= +\mathcal{T}^\mu{}_\rho\mathcal{T}^\nu{}_\sigma\overline{\Psi}iS^{\rho\sigma}\gamma_5\Psi \; . \end{aligned} \tag{40.51}$$

We have $C^{-1}\overline{\Psi}A\Psi C = \overline{\Psi}\mathcal{C}^{-1}A^{\rm T}\mathcal{C}\Psi$, and $\mathcal{C}^{-1}(\gamma^\mu\gamma^\nu)^{\rm T}\mathcal{C} = \mathcal{C}^{-1}\gamma^{\nu\rm T}\gamma^{\mu\rm T}\mathcal{C} = (\mathcal{C}^{-1}\gamma^{\nu\rm T}\mathcal{C})(\mathcal{C}^{-1}\gamma^{\mu\rm T}\mathcal{C})$ $= (-\gamma^\nu)(-\gamma^\mu) = \gamma^\nu\gamma^\mu = -\gamma^\mu\gamma^\nu$ for $\mu \neq \nu$. Using $\mathcal{C}^{-1}\gamma_5^{\rm T}\mathcal{C} = \gamma_5$, we have $\mathcal{C}^{-1}(\gamma^\mu\gamma^\nu\gamma_5)^{\rm T}\mathcal{C} = -\gamma^\mu\gamma^\nu\gamma_5$ for $\mu \neq \nu$. Therefore

$$\begin{aligned} C^{-1}\overline{\Psi}S^{\mu\nu}\Psi C &= -\overline{\Psi}S^{\mu\nu}\Psi \; , \\ C^{-1}\overline{\Psi}iS^{\mu\nu}\gamma_5\Psi C &= -\overline{\Psi}iS^{\mu\nu}\gamma_5\Psi \; . \end{aligned} \tag{40.52}$$

Since both are odd under C, both must vanish for a Majorana field. Under CPT, we have $(CPT)^{-1}\overline{\Psi}S^{\mu\nu}\Psi CPT = (\mathcal{TP})^\mu{}_\rho(\mathcal{TP})^\nu{}_\sigma\overline{\Psi}S^{\mu\nu}\Psi$. Since $(\mathcal{TP})^\mu{}_\rho = -\delta^\mu{}_\rho$, we see that $(CPT)^{-1}\overline{\Psi}S^{\mu\nu}\Psi CPT = \overline{\Psi}S^{\mu\nu}\Psi$. The same applies to $\overline{\Psi}S^{\mu\nu}i\gamma_5\Psi$.

41 LSZ Reduction for Spin-One-Half Particles

42 The Free Fermion Propagator

42.1) Multiply eq. (42.12) for $S(x-y)$ on the right by $\mathcal{C}$ and take the transpose. In problem 39.4d we showed that $[(-\not{p}+m)\mathcal{C}]^{\mathrm{T}} = (-\not{p}-m)\mathcal{C}$. If we then take $p \to -p$ and $x \leftrightarrow y$, we get $-S(y-x)\mathcal{C}$. Since $\mathcal{C}^{-1} = -\mathcal{C}$, this verifies eq. (42.24).

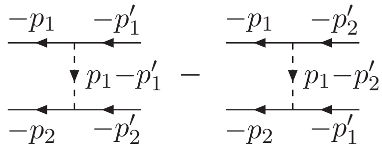
43 The Path Integral for Fermion Fields

44 Formal Development of Fermionic Path Integrals

45 The Feynman Rules for Dirac Fields

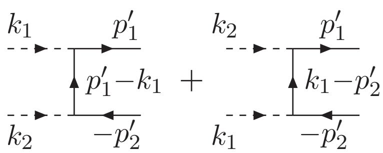
45.1) From section 40 we see that $\overline{\Psi}\Psi$ is even under P, T, ad C, while $i\overline{\Psi}\gamma_5\Psi$ is odd under P and T and even under C; in each case φ must have the same properties for the interaction term to be invariant.

45.2) For $e^+e^+ \to e^+e^+$, we have



$$i\mathcal{T}_{e^+e^+\to e^+e^+} = \tfrac{1}{i}(ig)^2\left[\frac{(\overline{v}_1 v_1')(\overline{v}_2 v_2')}{-t+M^2} - \frac{(\overline{v}_1 v_2')(\overline{v}_2 v_1')}{-u+M^2}\right].$$

For $\varphi\varphi \to e^+e^-$, we have



$$i\mathcal{T}_{\varphi\varphi\to e^+e^-} = \tfrac{1}{i}(ig)^2\,\overline{u}_1'\left[\frac{-\not{p}_1' + \not{k}_1 + m}{-t+m^2} + \frac{-\not{k}_1 + \not{p}_2' + m}{-u+m^2}\right]v_2' \,.$$

46 Spin Sums

47 Gamma Matrix Technology

47.1) If $\mu = \nu$, then $\gamma^\mu \gamma^\nu = \pm 1$, and we get $\text{Tr}\,\gamma_5 = 0$; if $\mu \neq \nu$, then $\gamma_5 \gamma^\mu \gamma^\nu \propto g^\rho \gamma^\sigma$, where μ, ν, ρ and σ are all different; in particular, $\rho \neq \sigma$, so $\text{Tr}[\gamma^\rho \gamma^\sigma] = 0$.

47.2) We have

$$
\begin{aligned}
\gamma^\mu \not{a} \not{b} \gamma_\mu &= (-\not{a}\gamma^\mu - 2a^\mu)(-\gamma_\mu \not{b} - 2b_\mu) \\
&= \not{a}\gamma^\mu \gamma_\mu \not{b} + 2\not{a}\not{b} + 2\not{a}\not{b} + 4(ab) \\
&= 4(ab) - (d{-}4)\not{a}\not{b}\,,
\end{aligned}
\tag{47.22}
$$

and

$$
\begin{aligned}
\gamma^\mu \not{a}\not{b}\not{c}\gamma_\mu &= (-\not{a}\gamma^\mu - 2a^\mu)\not{b}(-\gamma_\mu \not{c} - 2c_\mu) \\
&= \not{a}\gamma^\mu \not{b}\gamma_\mu \not{c} + 2\not{b}\not{a}\not{c} + 2\not{a}\not{c}\not{b} + 4(ac)\not{b} \\
&= (d{-}2)\not{a}\not{b}\not{c} + 2\not{b}\not{a}\not{c} + 2\not{a}\not{c}\not{b} + 4(ac)\not{b} \\
&= (d{-}2)\not{a}\not{b}\not{c} + 2\not{b}\not{a}\not{c} + 2[\not{a}\not{c} + 2(ac)]\not{b} \\
&= (d{-}2)\not{a}\not{b}\not{c} + 2\not{b}\not{a}\not{c} - 2\not{c}\not{a}\not{b} \\
&= (d{-}2)\not{a}\not{b}\not{c} + 2[-\not{a}\not{b} - 2(ab)]\not{c} - 2\not{c}\not{a}\not{b} \\
&= (d{-}4)\not{a}\not{b}\not{c} - 4(ab)\not{c} - 2\not{c}\not{a}\not{b} \\
&= (d{-}4)\not{a}\not{b}\not{c} + 2\not{c}\,[-2(ab) - \not{a}\not{b}] \\
&= 2\not{c}\not{b}\not{a} + (d{-}4)\not{a}\not{b}\not{c}\,.
\end{aligned}
\tag{47.23}
$$

48 Spin-Averaged Cross Sections

48.1)
```
(* Computes gamma matrix traces for the process 1+2 -> 3+4.
The format is, for example, tr[(-p1+m1*i).(-p2+m2*i)].
Note that * is used to multiply a matrix by a number,
and . is used to multiply two matrices.
Do not forget the . between matrices!  Do not forget to write mass terms as m*i !
(If you do, the program will give you an incorrect answer without warning.)
Terms with gamma matrices with contracted vector indices can be written as
Sum[tr[ ...  g[[mu]] ...  g[[mu]] ...  ],{mu,4}]//Simplify.
Do not use i or m or any other already named variable as an index! *)

(* the gamma matrices *)
i = IdentityMatrix[4];
g0 = {{ 0, 0, 1, 0},{0, 0, 0, 1},{ 1, 0, 0, 0},{ 0, 1, 0, 0}};
g1 = {{ 0, 0, 0, 1},{0, 0, 1, 0},{ 0,-1, 0, 0},{-1, 0, 0, 0}};
g2 = {{ 0, 0, 0,-I},{0, 0, I, 0},{ 0, I, 0, 0},{-I, 0, 0, 0}};
g3 = {{ 0, 0, 1, 0},{0, 0, 0,-1},{-1, 0, 0, 0},{ 0, 1, 0, 0}};
g5 = {{-1, 0, 0, 0},{0,-1, 0, 0},{ 0, 0, 1, 0},{ 0, 0, 0, 1}};

g = {g1,g2,g3,I*g0};

(* Particle energies in the CM frame *)
e1 = (s + m1^2 - m2^2)/(2 Sqrt[s]);
e2 = (s + m2^2 - m1^2)/(2 Sqrt[s]);
e3 = (s + m3^2 - m4^2)/(2 Sqrt[s]);
e4 = (s + m4^2 - m3^2)/(2 Sqrt[s]);

(* Magnitudes of 3-momenta in CM frame; k2=k1 and k4=k3 *)
k1 = (1/2)Sqrt[s-2(m1^2+m2^2)+(m1^2-m2^2)^2/s];
k3 = (1/2)Sqrt[s-2(m3^2+m4^2)+(m3^2-m4^2)^2/s];

(* 4-momenta dotted into gamma matrices; th is the CM scattering angle *)
p1 = -e1*g0 + k1*g3;
p2 = -e2*g0 - k1*g3;
p3 = -e3*g0 + k3*(Sin[th]*g1 + Cos[th]*g3);
p4 = -e4*g0 - k3*(Sin[th]*g1 + Cos[th]*g3);

(* Helicity 4-vectors dotted into gamma matrices; MASSIVE PARTICLES ONLY *)
h1 = (-k1*g0 + e1*g3)/m1;
h2 = (-k1*g0 - e2*g3)/m2;
h3 = (-k3*g0 + e3*(Sin[th]*g1 + Cos[th]*g3))/m3;
h4 = (-k3*g0 - e4*(Sin[th]*g1 + Cos[th]*g3))/m4;

(* ac = th in terms of the Mandelstam variables s and t *)
ac = ArcCos[(t - m1^2 - m3^2 + 2 e1 e3)/(2 k1 k3)];

(* Traces in terms of the Mandelstam variables s and t *)
tr[x_] := Sum[x[[j,j]],{j,4}]/.th->ac //Simplify
```

48.2) From eq. (45.23), we have

$$\mathcal{T}_{e^+e^-\to\varphi\varphi} = g^2\,\overline{v}_2\left[\frac{-\not p_1 + \not k_1' + m}{-t+m^2} + \frac{-\not p_1 + \not k_2' + m}{-u+m^2}\right]u_1\;, \tag{48.31}$$

where $t = -(p_1 - k_1')^2 = -(p_2 - k_2')^2$ and $u = -(p_1 - k_2')^2 = -(p_2 - k_1')^2$. We can use $-\not p_1 u_1 = m u_1$ to simplify this to

$$\mathcal{T} = g^2\,\overline{v}_2\left[\frac{\not k_1' + 2m}{-t+m^2} + \frac{\not k_2' + 2m}{-u+m^2}\right]u_1\;. \tag{48.32}$$

We then have

$$\overline{\mathcal{T}} = g^2\,\overline{u}_1\left[\frac{\not k_1' + 2m}{-t+m^2} + \frac{\not k_2' + 2m}{-u+m^2}\right]v_2\;. \tag{48.33}$$

Therefore

$$\begin{aligned}
|\mathcal{T}|^2 = {} & + \frac{g^4}{(m^2-t)^2}\,\mathrm{Tr}\Big[(v_2\overline{v}_2)(\not k_1' + 2m)(u_1\overline{u}_1)(\not k_1' + 2m)\Big] \\
& + \frac{g^4}{(m^2-u)^2}\,\mathrm{Tr}\Big[(v_2\overline{v}_2)(\not k_2' + 2m)(u_1\overline{u}_1)(\not k_2' + 2m)\Big] \\
& + \frac{g^4}{(m^2-t)(m^2-u)}\,\mathrm{Tr}\Big[(v_2\overline{v}_2)(\not k_1' + 2m)(u_1\overline{u}_1)(\not k_2' + 2m)\Big] \\
& + \frac{g^4}{(m^2-t)(m^2-u)}\,\mathrm{Tr}\Big[(v_2\overline{v}_2)(\not k_2' + 2m)(u_1\overline{u}_1)(\not k_1' + 2m)\Big]\;.
\end{aligned} \tag{48.34}$$

Averaging over the initial spins, we get

$$\langle|\mathcal{T}|^2\rangle = g^4\left[\frac{\langle\Phi_{tt}\rangle}{(m^2-t)^2} + \frac{\langle\Phi_{uu}\rangle}{(m^2-u)^2} + \frac{\langle\Phi_{tu}\rangle + \langle\Phi_{ut}\rangle}{(m^2-t)(m^2-u)}\right], \tag{48.35}$$

where

$$\begin{aligned}
\langle\Phi_{tt}\rangle &= \tfrac14\mathrm{Tr}\Big[(-\not p_2 - m)(\not k_1' + 2m)(-\not p_1 + m)(\not k_1' + 2m)\Big]\;, \\
\langle\Phi_{uu}\rangle &= \tfrac14\mathrm{Tr}\Big[(-\not p_2 - m)(\not k_2' + 2m)(-\not p_1 + m)(\not k_2' + 2m)\Big]\;, \\
\langle\Phi_{tu}\rangle &= \tfrac14\mathrm{Tr}\Big[(-\not p_2 - m)(\not k_1' + 2m)(-\not p_1 + m)(\not k_2' + 2m)\Big]\;, \\
\langle\Phi_{ut}\rangle &= \tfrac14\mathrm{Tr}\Big[(-\not p_2 - m)(\not k_2' + 2m)(-\not p_1 + m)(\not k_1' + 2m)\Big]\;.
\end{aligned} \tag{48.36}$$

We have

$$\begin{aligned}
\langle\Phi_{tt}\rangle &= \tfrac14\mathrm{Tr}[\not p_2\not k_1'\not p_1\not k_1'] + \tfrac14 m^2\,\mathrm{Tr}[4\not p_1\not p_2 + 2\not p_1\not k_1' + 2\not p_1\not k_1' - 2\not p_2\not k_1' - 2\not p_2\not k_1' - \not k_1'\not k_1'] - m^2\,\mathrm{Tr}\,1 \\
&= 2(p_1k_1')(p_2k_1') - (p_1p_2)k_1'^{\,2} - m^2(4p_1p_2 + 4p_1k_1' - 4p_2k_1' - k_1'^{\,2}) - 4m^4 \\
&= \tfrac12(t{-}m^2{-}M^2)(u{-}m^2{-}M^2) - \tfrac12(s{-}2m^2)M^2 \\
&\qquad - m^2[4(m^2{-}\tfrac12 s) + 2(t{-}m^2{-}M^2) - 2(u{-}m^2{-}M^2) + M^2)] - 4m^4 \\
&= -\tfrac12[-tu + m^2(9t+u) + 7m^4 - 8m^2M^2 + M^4]
\end{aligned} \tag{48.37}$$

and

$$\begin{aligned}\langle\Phi_{tu}\rangle &= \tfrac{1}{4}\mathrm{Tr}[\not{p}_2\not{k}_1'\not{p}_1\not{k}_2'] + \tfrac{1}{4}m^2\,\mathrm{Tr}[4\not{p}_1\not{p}_2 + 2\not{p}_1\not{k}_1' + 2\not{p}_1\not{k}_2' - 2\not{p}_2\not{k}_1' - 2\not{p}_2\not{k}_2' - \not{k}_1'\not{k}_2'] - m^2\,\mathrm{Tr}\,1 \\ &= (p_1k_1')(p_2k_2') + (p_1k_2')(p_2k_1') - (p_1p_2)(k_1'k_2') \\ &\quad - m^2(4p_1p_2 + 2p_1k_1' + 2p_1k_2' - 2p_2k_1' - 2p_2k_2' - k_1'k_2') - 4m^4 \\ &= \tfrac{1}{4}(t-m^2-M^2)^2 + \tfrac{1}{4}(u-m^2-M^2)^2 - \tfrac{1}{4}(s-2m^2)(s-2M^2) \\ &\quad - m^2[4(m^2-\tfrac{1}{2}s) - (M^2-\tfrac{1}{2}s)] - 4m^4 \\ &= -\tfrac{1}{2}[tu + 3m^2(t+u) + 9m^4 - 8m^2M^2 - M^4]\,. \end{aligned} \tag{48.38}$$

The extra factor of one-half compared to the crossing-related process $e^-\varphi \to e^-\varphi$ arises because we are summing (rather than averaging) over the final electron spin in the latter case.

48.3) From eq. (45.24), we have

$$\mathcal{T}_{e^-e^-\to e^-e^-} = g^2\left[\frac{(\overline{u}_1'u_1)(\overline{u}_2'u_2)}{-t+M^2} - \frac{(\overline{u}_2'u_1)(\overline{u}_1'u_2)}{-u+M^2}\right], \tag{48.39}$$

where $t = -(p_1-p_1')^2 = -(p_2-p_2')^2$ and $u = -(p_1-p_2')^2 = -(p_1-k_1')^2$. We also have

$$\overline{\mathcal{T}} = g^2\left[\frac{(\overline{u}_1u_1')(\overline{u}_2u_2')}{-t+M^2} - \frac{(\overline{u}_1u_2')(\overline{u}_2u_1')}{-u+M^2}\right]. \tag{48.40}$$

Therefore

$$|\mathcal{T}|^2 = g^4\left[\frac{\Phi_{ss}}{(M^2-s)^2} - \frac{\Phi_{st}+\Phi_{ts}}{(M^2-s)(M^2-t)} + \frac{\Phi_{tt}}{(M^2-t)^2}\right], \tag{48.41}$$

where

$$\begin{aligned}\Phi_{tt} &= \mathrm{Tr}\Big[u_1\overline{u}_1u_1'\overline{u}_1'\Big]\,\mathrm{Tr}\Big[u_2\overline{u}_2u_2'\overline{u}_2'\Big]\,, \\ \Phi_{uu} &= \mathrm{Tr}\Big[u_1\overline{u}_1u_2'\overline{u}_2'\Big]\,\mathrm{Tr}\Big[u_2\overline{u}_2u_1'\overline{u}_1'\Big]\,, \\ \Phi_{tu} &= \mathrm{Tr}\Big[u_1\overline{u}_1u_2'\overline{u}_2'u_2\overline{u}_2u_1'\overline{u}_1'\Big]\,, \\ \Phi_{ut} &= \mathrm{Tr}\Big[u_1\overline{u}_1u_1'\overline{u}_1'u_2\overline{u}_2u_2'\overline{u}_2'\Big]\,. \end{aligned} \tag{48.42}$$

Averaging over initial spins and summing over final spins, we get

$$\langle\Phi_{tt}\rangle = \tfrac{1}{4}\mathrm{Tr}\Big[(-\not{p}_1+m)(-\not{p}_1'+m)\Big]\,\mathrm{Tr}\Big[(-\not{p}_2+m)(-\not{p}_2'+m)\Big]\,, \tag{48.43}$$

$$\langle\Phi_{uu}\rangle = \tfrac{1}{4}\mathrm{Tr}\Big[(-\not{p}_1+m)(-\not{p}_2'+m)\Big]\,\mathrm{Tr}\Big[(-\not{p}_2+m)(-\not{p}_1'+m)\Big]\,, \tag{48.44}$$

$$\langle\Phi_{tu}\rangle = \tfrac{1}{4}\mathrm{Tr}\Big[(-\not{p}_1+m)(-\not{p}_2'+m)(-\not{p}_2+m)(-\not{p}_1'+m)\Big]\,, \tag{48.45}$$

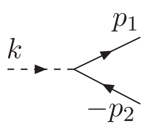
$$\langle\Phi_{ut}\rangle = \tfrac{1}{4}\mathrm{Tr}\Big[(-\not{p}_1+m)(-\not{p}_1'+m)(-\not{p}_2+m)(-\not{p}_2'+m)\Big]\,. \tag{48.46}$$

We get $\langle\Phi_{uu}\rangle$ from $\langle\Phi_{tt}\rangle$ and $\langle\Phi_{ut}\rangle$ from $\langle\Phi_{tu}\rangle$ by swapping $p_1' \leftrightarrow p_2'$, which is equivalent to swapping $t \leftrightarrow u$. Computing the traces, we find

$$\langle\Phi_{tt}\rangle = (t-4m^2)^2 \ , \tag{48.47}$$

$$\langle\Phi_{tu}\rangle = -\tfrac{1}{2}tu + 2m^2 s \ . \tag{48.48}$$

48.4) a)



For notational convenience we omit primes on the final momenta. The amplitude is then $\mathcal{T} = g\overline{u}_1 v_2$, and so $\overline{\mathcal{T}} = g\overline{v}_2 u_1$, $|\mathcal{T}|^2 = g^2\,\mathrm{Tr}[u_1\overline{u}_1 v_2\overline{v}_2]$, $\langle|\mathcal{T}|^2\rangle = g^2\,\mathrm{Tr}[(-\not{p}_1+m)(-\not{p}_2-m)] = g^2(-4p_1p_2-4m^2) = 2g^2(M^2-4m^2)$.

Using our results from problem 11.1b, we have $\Gamma = (\langle|\mathcal{T}|^2\rangle/16\pi M)(1-4m^2/M^2)^{1/2} = (g^2M/8\pi)(1-4m^2/M^2)^{3/2}$.

b) From eq. (38.28), with spin quantized along the x-axis we have

$$\begin{aligned} u_1\overline{u}_1 &= \tfrac{1}{2}(1-s_1\gamma_5\not{x})(-\not{p}_1+m) \ , \\ v_2\overline{v}_2 &= \tfrac{1}{2}(1-s_2\gamma_5\not{x})(-\not{p}_2-m) \ , \end{aligned} \tag{48.49}$$

and we take $\mathbf{p}_1 = -\mathbf{p}_2 = p\hat{\mathbf{z}}$ with $p = \frac{1}{2}M(1-4m^2/M^2)^{1/2}$. Thus we have

$$\begin{aligned} |\mathcal{T}|^2 &= g^2\,\mathrm{Tr}[u_1\overline{u}_1 v_2\overline{v}_2] \\ &= \tfrac{1}{4}g^2\,\mathrm{Tr}[(1-s_1\gamma_5\not{x})(-\not{p}_1+m)(1-s_2\gamma_5\not{x})(-\not{p}_2-m)] \ . \end{aligned} \tag{48.50}$$

Since a trace with a single γ_5 and three or fewer gamma matrices vanishes, we have

$$|\mathcal{T}|^2 = \tfrac{1}{4}g^2\,\mathrm{Tr}[(-\not{p}_1+m)(-\not{p}_2-m) + s_1s_2(\gamma_5\not{x})(-\not{p}_1+m)(\gamma_5\not{x})(-\not{p}_2-m)] \ . \tag{48.51}$$

Using $\gamma_5\not{a} = -\not{a}\gamma_5$ and $\gamma_5^2 = 1$, we have

$$|\mathcal{T}|^2 = \tfrac{1}{4}g^2\,\mathrm{Tr}[(-\not{p}_1+m)(-\not{p}_2-m) + s_1s_2\not{x}(-\not{p}_1-m)\not{x}(-\not{p}_2-m)] \ . \tag{48.52}$$

Then using $\not{a}\not{b} = -\not{b}\not{a} - 2ab$ along with $xp_1 = xp_2 = 0$ and $\not{x}\not{x} = -x^2 = -1$, we have

$$\begin{aligned} |\mathcal{T}|^2 &= \tfrac{1}{4}g^2\,\mathrm{Tr}[(-\not{p}_1+m)(-\not{p}_2-m) - s_1s_2(-\not{p}_1+m)(-\not{p}_2-m)] \\ &= \tfrac{1}{4}g^2(1+s_1s_2)\,\mathrm{Tr}[(-\not{p}_1+m)(-\not{p}_2-m)] \\ &= \tfrac{1}{4}g^2(1+s_1s_2)(-4p_1p_2-4m^2) \\ &= \tfrac{1}{2}g^2(1+s_1s_2)(M^2-4m^2) \ . \end{aligned} \tag{48.53}$$

This vanishes if $s_1 = -s_2$ or if $M = 2m$. Reason: since $\overline{\Psi}\Psi$ has even parity, so must φ. An electron-positron pair with orbital angular momentum ℓ has parity $-(-1)^\ell$. Thus ℓ must be odd. A particle with zero three-momentum cannot have nonzero orbital angular momentum, so $|\mathcal{T}|^2$ vanishes if $M = 2m$. Also, since the initial particle has spin zero, the total angular momentum must be zero. Thus there must be spin angular momentum to cancel the orbital angular momentum, and so the spins must be aligned; thus $|\mathcal{T}|^2$ vanishes if the spins are opposite.

c) For helicities s_1 and s_2, we have

$$\begin{aligned} p_1 &= (E,0,0,+p)\ , \\ p_2 &= (E,0,0,-p)\ , \\ z_1 &= (p,0,0,+E)/m\ , \\ z_2 &= (p,0,0,-E)/m\ , \end{aligned} \tag{48.54}$$

with $E = \frac{1}{2}M$ and $p = \frac{1}{2}M(1-4m^2/M^2)^{1/2}$. We have $z_1p_1 = z_2p_2 = 0$, and so

$$\begin{aligned} |\mathcal{T}|^2 &= \tfrac{1}{4}g^2\,\mathrm{Tr}[(1-s_1\gamma_5\not{z}_1)(-\not{p}_1+m)(1-s_2\gamma_5\not{z}_2)(-\not{p}_2-m)] \\ &= \tfrac{1}{4}g^2\,\mathrm{Tr}[(-\not{p}_1+m)(-\not{p}_2-m)+s_1s_2\not{z}_1(-\not{p}_1-m)\not{z}_2(-\not{p}_2-m)] \\ &= \tfrac{1}{4}g^2\,\mathrm{Tr}[(-\not{p}_1+m)(-\not{p}_2-m)+s_1s_2\not{z}_1(-\not{p}_1-m)\not{z}_2(-\not{p}_2-m)] \\ &= -g^2(p_1p_2+m^2)+g^2s_1s_2[(z_1p_2)(z_2p_1)-(z_1z_2)(p_1p_2)+m^2z_1z_2]\ . \end{aligned} \tag{48.55}$$

From eq. (48.54), we see that $z_1p_2 = z_2p_1 = -2Ep/m$ and $z_1z_2 = p_1p_2/m^2 = -(E^2+p^2)/m^2$. Plugging these in, we find

$$|\mathcal{T}|^2 = \tfrac{1}{2}g^2(1+s_1s_2)(M^2-4m^2)\ . \tag{48.56}$$

There can be no orbital angular momentum parallel to the linear momentum, and so the z component of the spin angular momentum must vanish. The total spin along the $\hat{z}$ axis is $s_1 - s_2$, and so the helicities must be the same to get a nonzero $|\mathcal{T}|^2$. Parity again explains why $|\mathcal{T}|^2 = 0$ if $M = 2m$.

d) Now the amplitude is $\mathcal{T} = ig\overline{u}_1\gamma_5 v_2$, and so $\overline{\mathcal{T}} = ig\overline{v}_2\gamma_5 u_1$, $|\mathcal{T}|^2 = -g^2\,\mathrm{Tr}[v_2\overline{v}_2\gamma_5 u_1\overline{u}_1\gamma_5]$, $\langle|\mathcal{T}|^2\rangle = -g^2\,\mathrm{Tr}[(-\not{p}_2-m)\gamma_5(-\not{p}_1+m)\gamma_5] = -g^2\,\mathrm{Tr}[(-\not{p}_2-m)(\not{p}_1+m)] = -g^2(4p_1p_2-4m^2) = 2g^2M^2$. This is larger by a factor of $M^2/(M^2-4m^2)$. It is larger because $i\overline{\Psi}\gamma_5\Psi$ has odd parity, and therefore so must φ. Thus the orbital angular momentum of the electron-positron pair must be even, and in particular must be zero, since $\ell = 2$ or larger could not be cancelled by spin. With zero orbital angular momentum, $|\mathcal{T}|^2$ need not vanish for zero electron three-momentum, leading to a larger decay rate.

e) Redoing part (b) yields

$$\begin{aligned} |\mathcal{T}|^2 &= -\tfrac{1}{4}g^2\,\mathrm{Tr}[(1-s_1\gamma_5\not{z})(-\not{p}_1+m)\gamma_5(1-s_2\gamma_5\not{z})(-\not{p}_2-m)-\gamma_5] \\ &= -\tfrac{1}{4}g^2\,\mathrm{Tr}[(1-s_1\gamma_5\not{z})(-\not{p}_1+m)(1+s_2\gamma_5\not{z})(\not{p}_2-m)] \end{aligned} \tag{48.57}$$

Comparing with the second line of eq. (48.50), we see that eq. (48.57) has an extra overall minus sign, $s_2 \to -s_2$, and $p_2 \to -p_2$. Therefore, comparing with the third line of eq. (48.53), we get

$$\begin{aligned} |\mathcal{T}|^2 &= -\tfrac{1}{4}g^2(1-s_1s_2)(4p_1p_2-4m^2) \\ &= \tfrac{1}{2}g^2(1-s_1s_2)M^2\ . \end{aligned} \tag{48.58}$$

This vanishes if $s_1 = s_2$. We know the electron-positron pair has even orbital angular momentum, and the total angular momentum must be zero. The only possibility is $\ell = 0$ and $s = s_1 + s_2 = 0$, so $|\mathcal{T}|^2$ vanishes if $s_1 = s_2$.

Redoing part (c) yields the same changes. Therefore, comparing with the last line of eq. (48.55) yields

$$\begin{aligned} |\mathcal{T}|^2 &= g^2(-p_1p_2+m^2) + g^2 s_1 s_2[-(z_1p_2)(z_2p_1) + (z_1z_2)(p_1p_2) + m^2 z_1 z_2] \\ &= \tfrac{1}{2} g^2 (1+s_1s_2) M^2 \; . \end{aligned} \tag{48.59}$$

As in part (c), there can be no orbital angular momentum parallel to the linear momentum, and so the z component of the spin angular momentum must vanish. The total spin along the $\hat{z}$ axis is $s_1 - s_2$, and so the helicities must be the same to get a nonzero $|\mathcal{T}|^2$.

48.5) Let $g \equiv c_1 G_{\rm F} f_\pi$; the vertex factor is then $(ig)(ik_\mu)\gamma^\mu(1-\gamma_5) = -g\not{k}(1-\gamma_5)$, where k is the four-momentum of the pion. Thus we have $i\mathcal{T} = -g\overline{u}_1\not{k}(1-\gamma_5)v_2$, where p_1 is the muon momentum and p_2 is the antineutrino momentum. We now use $\not{k} = \not{p}_1 + \not{p}_2$, $\not{p}_2(1-\gamma_5) = (1+\gamma_5)\not{p}_2$, $\overline{u}_1\not{p}_1 = -m_\mu \overline{u}_1$, and $\not{p}_2 v_2 = 0$ to get $\mathcal{T} = -igm_\mu\overline{u}_1(1-\gamma_5)v_2$. Then $\overline{\mathcal{T}} = +igm_\mu\overline{v}_2(1+\gamma_5)u_1$, and $|\mathcal{T}|^2 = g^2m_\mu^2\,{\rm Tr}[u_1\overline{u}_1(1-\gamma_5)v_2\overline{v}_2(1+\gamma_5)]$. Summing over final spins yields

$$\begin{aligned} \langle|\mathcal{T}|^2\rangle &= g^2m_\mu^2 {\rm Tr}[(-\not{p}_1+m_\mu)(1-\gamma_5)(-\not{p}_2)(1+\gamma_5)] \\ &= g^2m_\mu^2\,{\rm Tr}[(-\not{p}_1+m_\mu)(-\not{p}_2)(1+\gamma_5)(1+\gamma_5)] \\ &= 2g^2m_\mu^2\,{\rm Tr}[(-\not{p}_1+m_\mu)(-\not{p}_2)(1+\gamma_5)] \\ &= 2g^2m_\mu^2\,{\rm Tr}[\not{p}_1\not{p}_2] \\ &= 2g^2m_\mu^2\,(-4p_1p_2) \\ &= 4g^2m_\mu^2\,[-(p_1+p_2)^2 + p_1^2 + p_2^2] \\ &= 4g^2m_\mu^2\,(-k^2 + p_1^2 + p_2^2) \\ &= 4g^2m_\mu^2\,(m_\pi^2 - m_\mu^2 + 0) \; . \end{aligned} \tag{48.60}$$

We then have $\Gamma = \langle|\mathcal{T}|^2\rangle|\mathbf{p}_1|/8\pi m_\pi^2$, and $|\mathbf{p}_1| = (m_\pi^2 - m_\mu^2)/2m_\pi^2$, so

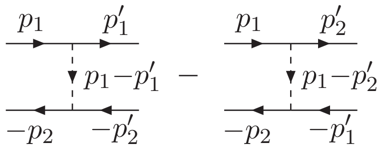
$$\Gamma = \frac{g^2m_\mu^2}{4\pi m_\pi^3}(m_\pi^2 - m_\mu^2)^2 \; . \tag{48.61}$$

Using $\Gamma = \hbar c/c\tau = (1.973 \times 10^{-14}\,{\rm GeV\,cm})/(2.998 \times 10^{10}\,{\rm cm/s})(2.603 \times 10^{-8}\,{\rm s}) = 2.528 \times 10^{-17}\,{\rm GeV}$, we find $g = 1.058 \times 10^{-6}\,{\rm GeV}$, and so $f_\pi = 93.14$; after including electromagnetic loop corrections, the result drops slightly to $f_\pi = 92.4$.

49 The Feynman Rules for Majorana Fields

49.1) a) The hermitian conjugate term is $\sqrt{2}eE_{\rm L}\overline{\Psi}P_{\rm R}X + \sqrt{2}eE_{\rm R}\overline{\Psi}P_{\rm L}X$.

b) The contributing diagrams are



where the exchanged scalar can be either $E_{\rm L}$ or $E_{\rm R}$. The arrows are drawn so that we use our standard conventions for the Dirac electron-positron field; reversing the lower arrow on each diagram and comparing the two shows that the relative sign is negative. The amplitude is

$$\mathcal{T} = 2e^2\biggl[\frac{(\overline{u}_1'P_{\rm L}u_1)(\overline{v}_2P_{\rm R}v_2')}{M_{\rm L}^2 - t} - \frac{(\overline{u}_2'P_{\rm L}u_1)(\overline{v}_2P_{\rm R}v_1')}{M_{\rm L}^2 - u} + \frac{(\overline{u}_1'P_{\rm R}u_1)(\overline{v}_2P_{\rm L}v_2')}{M_{\rm R}^2 - t} - \frac{(\overline{u}_2'P_{\rm R}u_1)(\overline{v}_2P_{\rm L}v_1')}{M_{\rm R}^2 - u}\biggr]. \tag{49.10}$$

c) In the limit $|t|, |u| \ll M^2 = M_{\rm L}^2 = M_{\rm R}^2$, we have

$$\mathcal{T} = \frac{2e^2}{M^2}\Bigl[(\overline{u}_1'P_{\rm L}u_1)(\overline{v}_2P_{\rm R}v_2') - (\overline{u}_2'P_{\rm L}u_1)(\overline{v}_2P_{\rm R}v_1') + (\overline{u}_1'P_{\rm R}u_1)(\overline{v}_2P_{\rm L}v_2') - (\overline{u}_2'P_{\rm R}u_1)(\overline{v}_2P_{\rm L}v_1')\Bigr]. \tag{49.11}$$

To facilitate squaring and summing over spins, it will be convenient to rewrite everything in terms of u spinors by using $\overline{v}P_{\rm L,R}v' = (\overline{v}P_{\rm L,R}v')^{\rm T} = v'^{\rm T}P_{\rm L,R}^{\rm T}\overline{v}^{\rm T} = \overline{u}'\mathcal{C}^{-1}P_{\rm L,R}^{\rm T}\mathcal{C}^{-1}u = -\overline{u}'P_{\rm L,R}u$, where the last equality follows from $\mathcal{C}^{-1} = -\mathcal{C}$ and $\mathcal{C}^{-1}\gamma_5\mathcal{C} = \gamma_5$. We then have

$$\begin{aligned}\mathcal{T} &= -\frac{2e^2}{M^2}\Bigl[(\overline{u}_1'P_{\rm L}u_1)(\overline{u}_2'P_{\rm R}u_2) - (\overline{u}_2'P_{\rm L}u_1)(\overline{u}_1'P_{\rm R}u_2) + (\overline{u}_1'P_{\rm R}u_1)(\overline{u}_2'P_{\rm L}u_2) - (\overline{u}_2'P_{\rm R}u_1)(\overline{u}_1'P_{\rm L}u_2)\Bigr] \\ &= -\frac{2e^2}{M^2}\Bigl[t_{\rm L} - u_{\rm L} + t_{\rm R} - u_{\rm R}\Bigr]\,. \end{aligned} \tag{49.12}$$

Barring, we get

$$\begin{aligned}\overline{\mathcal{T}} &= -\frac{2e^2}{M^2}\Bigl[(\overline{u}_1P_{\rm R}u_1')(\overline{u}_2P_{\rm L}u_2') - (\overline{u}_1P_{\rm R}u_2')(\overline{u}_2P_{\rm L}u_1') + (\overline{u}_1P_{\rm L}u_1')(\overline{u}_2P_{\rm R}u_2') - (\overline{u}_1P_{\rm L}u_2')(\overline{u}_2P_{\rm R}u_1)\Bigr] \\ &= -\frac{2e^2}{M^2}\Bigl[\overline{t}_{\rm L} - \overline{u}_{\rm L} + \overline{t}_{\rm R} - \overline{u}_{\rm R}\Bigr]\,. \end{aligned} \tag{49.13}$$

Then we have

$$|\mathcal{T}|^2 = \frac{4e^4}{M^4}\Bigl[\Bigl(\overline{t}_{\rm L}t_{\rm L} - \overline{t}_{\rm L}u_{\rm L} + \overline{t}_{\rm L}t_{\rm R} - \overline{t}_{\rm L}u_{\rm R} + (t \leftrightarrow u)\Bigr) + ({\rm L} \leftrightarrow {\rm R})\Bigr]. \tag{49.14}$$

We can write

$$\overline{t}_{\rm L}t_{\rm L} = {\rm Tr}[(u_1\overline{u}_1)P_{\rm R}(u_1'\overline{u}_1')P_{\rm L}]\,{\rm Tr}[(u_2\overline{u}_2)P_{\rm L}(u_2'\overline{u}_2')P_{\rm R}]\,, \tag{49.15}$$

$$\overline{t}_{\rm L}u_{\rm L} = {\rm Tr}[(u_1\overline{u}_1)P_{\rm R}(u_1'\overline{u}_1')P_{\rm R}(u_2\overline{u}_2)P_{\rm L}(u_2'\overline{u}_2')P_{\rm L}]\,, \tag{49.16}$$

$$\overline{t}_{\rm L}t_{\rm R} = {\rm Tr}[(u_1\overline{u}_1)P_{\rm R}(u_1'\overline{u}_1')P_{\rm R}]\,{\rm Tr}[(u_2\overline{u}_2)P_{\rm L}(u_2'\overline{u}_2')P_{\rm L}]\,, \tag{49.17}$$

$$\overline{t}_{\rm L}u_{\rm R} = {\rm Tr}[(u_1\overline{u}_1)P_{\rm R}(u_1'\overline{u}_1')P_{\rm L}(u_2\overline{u}_2)P_{\rm L}(u_2'\overline{u}_2')P_{\rm R}]\,. \tag{49.18}$$

Averaging over initial spins and summing over final spins, we get

$$\langle \bar t_{\rm L} t_{\rm L}\rangle = \tfrac14 {\rm Tr}[(-\not p_1)P_{\rm R}(-\not p_1'+m)P_{\rm L}]\,{\rm Tr}[(-\not p_2)P_{\rm L}(-\not p_2'+m)P_{\rm R}]\ , \tag{49.19}$$

$$\langle \bar t_{\rm L} u_{\rm L}\rangle = \tfrac14 {\rm Tr}[(-\not p_1)P_{\rm R}(-\not p_1'+m)P_{\rm R}(-\not p_2)P_{\rm L}(-\not p_2'+m)P_{\rm L}]\ , \tag{49.20}$$

$$\langle \bar t_{\rm L} t_{\rm R}\rangle = \tfrac14 {\rm Tr}[(-\not p_1)P_{\rm R}(-\not p_1'+m)P_{\rm R}]\,{\rm Tr}[(-\not p_2)P_{\rm L}(-\not p_2'+m)P_{\rm L}]\ , \tag{49.21}$$

$$\langle \bar t_{\rm L} u_{\rm R}\rangle = \tfrac14 {\rm Tr}[(-\not p_1)P_{\rm R}(-\not p_1'+m)P_{\rm L}(-\not p_2)P_{\rm L}(-\not p_2'+m)P_{\rm R}]\ , \tag{49.22}$$

where $m \equiv m_{\tilde\gamma}$ is the photino mass (and we neglect the mass of the electron). We use $\not p P_{\rm R} = P_{\rm L}\not p$, $P^2_{\rm L,R} = P_{\rm L,R}$, $P_{\rm R}P_{\rm L} = 0$, and the cyclic property of the trace to get

$$\begin{aligned}
\langle \bar t_{\rm L} t_{\rm L}\rangle &= \tfrac14 {\rm Tr}[(-\not p_1)(-\not p_1'+m)P_{\rm L}]\,{\rm Tr}[(-\not p_2)(-\not p_2'+m)P_{\rm R}] \\
&= \tfrac{1}{16}{\rm Tr}[\not p_1\not p_1']\,{\rm Tr}[\not p_2\not p_2'] \\
&= (p_1p_1')(p_2p_2') \\
&= \tfrac14(t-m^2)^2\ ,
\end{aligned} \tag{49.23}$$

$$\begin{aligned}
\langle \bar t_{\rm L} u_{\rm L}\rangle &= \tfrac14 {\rm Tr}[(-\not p_1)(-\not p_1'+m)(-\not p_2)P_{\rm L}(-\not p_2'+m)P_{\rm L}] \\
&= \tfrac14 m\,{\rm Tr}[(-\not p_1)(-\not p_1'+m)(-\not p_2)P_{\rm L}] \\
&= \tfrac18 m^2\,{\rm Tr}[\not p_1\not p_2] \\
&= -\tfrac12 m^2(p_1p_2) \\
&= \tfrac14 m^2 s\ ,
\end{aligned} \tag{49.24}$$

$$\langle \bar t_{\rm L} t_{\rm R}\rangle = 0\ , \tag{49.25}$$

$$\langle \bar t_{\rm L} u_{\rm R}\rangle = 0\ . \tag{49.26}$$

Plugging these into eq. (49.14), we get

$$\begin{aligned}
\langle |\mathcal{T}|^2\rangle &= \frac{e^4}{M^4}\Big[\Big((t-m^2)^2 - m^2 s + (t\leftrightarrow u)\Big) + ({\rm L}\leftrightarrow{\rm R})\Big] \\
&= \frac{2e^4}{M^4}\Big[(t-m^2)^2 + (u-m^2)^2 - 2m^2 s\Big]\ .
\end{aligned} \tag{49.27}$$

The cross section is

$$\frac{d\sigma}{d\Omega_{\rm CM}} = \frac{1}{64\pi^2 s}\,\frac{|\mathbf{p}_1'|}{|\mathbf{p}_1|}\,\langle |\mathcal{T}|^2\rangle\ , \tag{49.28}$$

with $|\mathbf{p}_1| = \frac12 s^{1/2}$ and $|\mathbf{p}_1'| = \frac12(s-4m^2)^{1/2}$. Also, $t-m^2 = 2p_1p_1' = -2E_1E_1' + 2|\mathbf{p}_1||\mathbf{p}_1'|\cos\theta = -\frac12 s + \frac12[s(s-4m^2)]^{1/2}\cos\theta$, and similarly $u-m^2 = -\frac12 s - \frac12[s(s-4m^2)]^{1/2}\cos\theta$. Plugging these in, we get

$$\frac{d\sigma}{d\Omega_{\rm CM}} = \frac{e^4}{64\pi^2 M^4}\,s(1-4m^2/s)^{3/2}\,(1+\cos^2\theta)\ . \tag{49.29}$$

The $(1-4m^2/s)^{3/2}$ threshold behavior is characteristic of a final state with orbital angular momentum $\ell = 1$, as we saw in problem 48.4. The $1+\cos^2\theta$ angular distribution is also characteristic of $\ell = 1$, since $1+\cos^2\theta \propto \sum_{m=-1,0,1}|Y_{1m}(\theta,\phi)|^2$; we say that this is a *p-wave process.*

50 Massless Particles and Spinor Helicity

50.1) a) $-\not{p} = \sum_s u_s(\mathbf{p})\overline{u}_s(\mathbf{p}) = u_+(\mathbf{p})\overline{u}_+(\mathbf{p}) + u_-(\mathbf{p})\overline{u}_-(\mathbf{p}) = |p\rangle[p| + |p]\langle p|$.

b) We have

$$\begin{aligned}
\overline{u}_+(\mathbf{p}')(-\not{k})u_+(\mathbf{p}) &= [p'|\Big(|k\rangle[k| + |k]\langle k|\Big)|p\rangle = 0 + [p'\,k]\langle k\,p\rangle\;, \\
\overline{u}_-(\mathbf{p}')(-\not{k})u_-(\mathbf{p}) &= \langle p'|\Big(|k\rangle[k| + |k]\langle k|\Big)|p] = \langle p'\,k\rangle[k\,p] + 0\;, \\
\overline{u}_+(\mathbf{p}')(-\not{k})u_-(\mathbf{p}) &= [p'|\Big(|k\rangle[k| + |k]\langle k|\Big)|p] = 0 + 0\;, \\
\overline{u}_-(\mathbf{p}')(-\not{k})u_+(\mathbf{p}) &= \langle p'|\Big(|k\rangle[k| + |k]\langle k|\Big)|p\rangle = 0 + 0\;.
\end{aligned} \tag{50.46}$$

50.2) a) We have

$$\begin{aligned}
-\phi_a\phi^*_{\dot a} &= -2\omega\begin{pmatrix} \sin^2(\tfrac{1}{2}\theta) & -\sin(\tfrac{1}{2}\theta)\cos(\tfrac{1}{2}\theta)e^{-i\phi} \\ -\sin(\tfrac{1}{2}\theta)\cos(\tfrac{1}{2}\theta)e^{+i\phi} & \cos^2(\tfrac{1}{2}\theta) \end{pmatrix} \\
&= +\omega\begin{pmatrix} -1+\cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{+i\phi} & -1-\cos\theta \end{pmatrix} \\
&= \begin{pmatrix} -p^0+p^3 & p^1 - ip^2 \\ p^1+ip^2 & -p^0-p^3 \end{pmatrix}.
\end{aligned} \tag{50.47}$$

b) From the solution to problem 38.1, with $\hat{\mathbf{p}} = \hat{\mathbf{z}}$, we have

$$\exp(i\eta\hat{\mathbf{p}}\cdot\mathbf{K}) = \begin{pmatrix} (\cosh\tfrac{1}{2}\eta) - (\sinh\tfrac{1}{2}\eta)\sigma_3 & 0 \\ 0 & (\cosh\tfrac{1}{2}\eta) + (\sinh\tfrac{1}{2}\eta)\sigma_3 \end{pmatrix}. \tag{50.48}$$

In the $\eta \to \infty$ limit, we have $\cosh\frac{1}{2}\eta \simeq \sinh\frac{1}{2}\eta \simeq (E/2m)^{1/2}$. Eq. (50.48) becomes

$$\begin{aligned}
\exp(i\eta\hat{\mathbf{p}}\cdot\mathbf{K}) &= (E/2m)^{1/2}\begin{pmatrix} 1-\sigma_3 & 0 \\ 0 & 1+\sigma_3 \end{pmatrix} \\
&= (2E/m)^{1/2}\begin{pmatrix} 0&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&0 \end{pmatrix}.
\end{aligned} \tag{50.49}$$

With $u_\pm(\mathbf{0})$ given by eq. (38.6), we get $u_+(\mathbf{p}) = (2E)^{1/2}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$ and $u_-(\mathbf{p}) = (2E)^{1/2}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$.

For $\theta = 0$, this agrees with eqs. (50.8), (50.9), and (50.13).

50.3) The left-hand side of eq. (50.36) is manifestly cyclically symmetric on $q \to r \to s \to q$. To see that it is anitsymmetric on (say) $q \leftrightarrow r$, we use $\langle a\,b\rangle = -\langle b\,a\rangle$ to get

$$\langle p\,r\rangle\langle q\,s\rangle + \langle p\,q\rangle\langle s\,r\rangle + \langle p\,s\rangle\langle r\,q\rangle = -\langle p\,r\rangle\langle s\,q\rangle - \langle p\,q\rangle\langle r\,s\rangle - \langle p\,s\rangle\langle q\,r\rangle\;, \tag{50.50}$$

and note that the right-hand side is minus the left-hand side of eq. (50.36). Thus it is completely antisymmetric on q, r, s, and linear in each of the corresponding twistors. Since each twistor has only two components, the result must be zero.

50.4) We use $P_{\rm L}(-\not p) = u_-(\mathbf{p})\overline{u}_-(\mathbf{p}) = |p]\langle p|$ and $-\not q = |q]\langle q| + |q\rangle[q|$, etc., to get

$$\begin{aligned} \tfrac{1}{2}(1-\gamma_5)\not p\not q\not r\not s &= |p]\langle p|\Big(|q]\langle q| + |q\rangle[q|\Big)\Big(|r]\langle r| + |r\rangle[r|\Big)\Big(|s]\langle s| + |s\rangle[s|\Big) \\ &= |p]\,\langle p\,q\rangle\,[q\,r]\,\langle r\,s\rangle\,[s|\;. \end{aligned} \tag{50.51}$$

Taking the trace gives $\mathrm{Tr}\,\tfrac{1}{2}(1-\gamma_5)\not p\not q\not r\not s = \langle p\,q\rangle\,[q\,r]\,\langle r\,s\rangle\,[s\,p]$. The traces are standard and yield $2(pq)(rs) - 2(pr)(qs) + 2(ps)(qr) + 2i\varepsilon^{\mu\nu\rho\sigma}p_\mu q_\nu r_\rho s_\sigma$.

50.5) a) These are all vectors, and we note that no nonzero four-vector is orthogonal to every massless four vector. Therefore it is enough to verify eqs. (50.38–42) contracted with an arbitrary massless four-vector q_μ. Then we set $-\not q = |q]\langle q| + |q\rangle[q|$, and use the usual inner products along with eqs. (50.20), (50.21), and (50.24). For example, to verify eq. (50.38), we use $\langle p|\not q|k] = -\langle p\,q\rangle\,[q\,k] = -[k\,q]\,\langle q\,p\rangle = [k|\not q|p\rangle$. To verify eq. (50.40), we use $\langle p|\not q|p] = -\langle p\,q\rangle\,[q\,p] = 2p^\mu q_\mu$.

b) Again this follows immediately from contracting all the gamma matrices with arbitrary massless four-vectors, and using $-\not q = |q]\langle q| + |q\rangle[q|$, followed by the usual inner products.

c) We have $\langle p|\gamma_\mu|q] = \overline{u}_-(\mathbf{p})\gamma^\mu u_-(\mathbf{q}) = \phi^*_{\dot a}\bar\sigma^{\dot a a}_\mu\kappa_a$. Contracting with $-\tfrac{1}{2}\gamma^\mu$ gives

$$\begin{aligned} -\tfrac{1}{2}\langle p|\gamma_\mu|q]\gamma^\mu &= -\tfrac{1}{2}\phi^*_{\dot a}\bar\sigma^{\dot a a}_\mu\kappa_a\begin{pmatrix} 0 & \sigma^\mu_{c\dot c} \\ \bar\sigma^{\mu\dot b b} & 0 \end{pmatrix} \\ &= -\tfrac{1}{2}\phi^*_{\dot a}\kappa_a\begin{pmatrix} 0 & \bar\sigma^{\dot a a}_\mu\sigma^\mu_{c\dot c} \\ \bar\sigma^{\dot a a}_\mu\bar\sigma^{\mu\dot b b} & 0 \end{pmatrix} \\ &= -\tfrac{1}{2}\phi^*_{\dot a}\kappa_a\begin{pmatrix} 0 & -2\delta^a{}_c\delta^{\dot a}{}_{\dot c} \\ -2\varepsilon^{\dot a\dot b}\varepsilon^{ab} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \kappa_c\phi^*_{\dot c} \\ \phi^{*\dot b}\kappa^b & 0 \end{pmatrix} \\ &= u_-(\mathbf{q})\overline{u}_-(\mathbf{p}) + u_+(\mathbf{p})\overline{u}_+(\mathbf{q}) \\ &= |q]\langle p| + |p\rangle[q|\;. \end{aligned} \tag{50.52}$$

We similarly get eq. (50.44), and then eq. (50.45) follows immediately.

51 Loop Corrections in Yukawa Theory

51.1) We rewrite eq. (45.2) in the shorthand notation

$$Z = \exp\Big(ig\varphi_x\delta_x G\bar{\delta}_x\Big)\exp\Big(i\bar{\eta}_y S_{yz}\eta_z\Big)\;, \tag{51.55}$$

where $\delta_x = \delta/\delta\eta(x)$, $\bar{\delta}_x = \delta/\delta\bar{\eta}(x)$, spinor indices are suppressed, and spatial indices are implicitly integrated; G is a spin matrix which is either 1 or $i\gamma_5$, depending on which version of the theory we consider. Also, since we are interested in the fermion loop, we have replaced $(1/i)\delta/\delta J(x)$ with an external field $\varphi(x)$. A single closed loop with n external φ lines will correspond to a term of the form $g^n\,\varphi_1\ldots\varphi_n\,\mathrm{Tr}\,S_{12}GS_{23}G\ldots S_{n-1,n}G$. This is what we would get from the Feynman rules without an extra -1 from the closed loop; the i from each vertex is cancelled by the $1/i$ from each internal fermion line. To see that we do get the -1, consider the case of $n=2$; the relevant factors are $i^4g^2\,\varphi_1\varphi_2\,(\delta_1G\bar{\delta}_1)(\delta_2G\bar{\delta}_2)(\bar{\eta}_xS_{xy}\eta_y)(\bar{\eta}_zS_{zw}\eta_w)$. We pull the $\delta_2G\bar{\delta}_2$ through the $\bar{\eta}_x$, and allow it to act on $\eta_y\bar{\eta}_z$; we have $\delta_2\bar{\delta}_2\eta_y\bar{\eta}_z = -\delta_2\eta_y\bar{\delta}_2\bar{\eta}_z = -\delta_{2y}\delta_{2z}$. We now have $-g^2\,\varphi_1\varphi_2\,(\delta_1G\bar{\delta}_1)\bar{\eta}_x(S_{x2}GS_{2w})\eta_w = -g^2\,\varphi_1\varphi_2\,\mathrm{Tr}\,S_{12}GS_{21}G$. This pattern persists at larger n; we get an extra minus sign from the first vertex that we pull through, and then the remaining ones all give plus signs. (Note that we are only considering terms corresponding to a single closed fermion loop.)

51.2) For $p=p'=0$, we have $\tilde{N}=\tilde{N}_0=m^2\gamma_5$ and $D=D_0=(1{-}x_3)m^2+x_3M^2$. Since there is no dependence on x_1 and x_2, $\int dF_3$ becomes $2\int_0^1 dx_3\,(1{-}x_3)$. Performing the integral over x_3 yields

$$i\mathbf{V}_Y(0,0) = -Z_g g\gamma_5 + \frac{g^3}{8\pi^2}\left[\frac{1}{\varepsilon}+\frac{1}{2}-\frac{m^2\ln(M/m)}{M^2-m^2}-\ln(M/\mu)\right]\gamma_5\;. \tag{51.56}$$

Requiring this to equal $-g\gamma_5$ yields

$$Z_g = 1+\frac{g^2}{8\pi^2}\left[\frac{1}{\varepsilon}+\frac{1}{2}-\frac{m^2\ln(M/m)}{M^2-m^2}-\ln(M/\mu)\right]. \tag{51.57}$$

This then yields

$$\mathbf{V}_Y(p',p) = \left\{1+\frac{g^2}{8\pi^2}\left[-\frac{3}{4}+\frac{m^2\ln(M/m)}{M^2-m^2}+\int dF_3\left(-\tfrac{1}{2}\ln(D/M^2)+\frac{\tilde{N}}{4D}\right)\right]\right\}ig\gamma_5\;, \tag{51.58}$$

which is finite and independent of μ.

51.3) Consider the scalar propagator. The diagrams in fig. 51.1 all contribute. The fermion loop now has a factor of $\mathrm{Tr}\,\tilde{S}(\not\ell{+}\not k)\tilde{S}(\not\ell)$ instead of $\mathrm{Tr}\,\tilde{S}(\not\ell{+}\not k)i\gamma_5\tilde{S}(\not\ell)i\gamma_5$; this changes N, given by eq. (51.14), from $(\ell{+}k)\ell+m^2$ to $(\ell{+}k)\ell-m^2$. We then find $\Pi_{\Psi\,\mathrm{loop}}(k^2) = -(g^2/4\pi^2\varepsilon)(k^2+6m^2)$. The φ loop with the φ^4 vertex obviously gives the same result as before. There is now also a new φ loop diagram due to the φ^3 vertex; it is given by

$$\Pi_{\varphi^3\,\mathrm{loop}} = \tfrac{1}{2}\kappa^2\tilde{\mu}^{\varepsilon}\int\frac{d^d\ell}{(2\pi)^d}\,\tilde{\Delta}((\ell{+}k)^2)\tilde{\Delta}(\ell^2)\;, \tag{51.59}$$

where $d = 4-\varepsilon$. The divergent part is $\Pi_{\varphi^3\,\rm loop} = \kappa^2/16\pi^2\varepsilon$ (note that κ has dimensions of mass for $d=4$). Choosing Z_φ and Z_M to cancel the divergences yields

$$Z_\varphi = 1 - \frac{g^2}{4\pi^2}\frac{1}{\varepsilon}\,, \tag{51.60}$$

$$Z_M = 1 + \left(\frac{\lambda}{16\pi^2} - \frac{3g^2}{2\pi^2}\frac{m^2}{M^2} + \frac{\kappa^2}{16\pi^2}\right)\frac{1}{\varepsilon}\,. \tag{51.61}$$

The loop correction to the fermion propagator now has a factor of $S(\not{p}+\not{\ell})$ rather than $i\gamma_5\tilde{S}(\not{p}+\not{\ell})i\gamma_5$; this changes N, given by eq. (51.30), from $\not{q}+(1-x)\not{p}+m$ to $\not{q}+(1-x)\not{p}-m$. Thus we get $\Sigma_{1\,\rm loop} = -(g^2/16\pi^2\varepsilon)(\not{p}-2m)$, and so

$$Z_\Psi = 1 - \frac{g^2}{16\pi^2}\frac{1}{\varepsilon}\,, \tag{51.62}$$

$$Z_m = 1 + \frac{g^2}{8\pi^2}\frac{1}{\varepsilon}\,. \tag{51.63}$$

For the loop correction to the Yukawa coupling, again a factor of $i\gamma_5$ is removed from each vertex. We then find that $N = q^2 + \tilde{N}$ instead of eq. (51.45); $\tilde{N}$ is different, but does not contribute to the divergent part. So

$$Z_g = 1 + \frac{g^2}{8\pi^2}\frac{1}{\varepsilon}\,. \tag{51.64}$$

There is also a new diagram where the external φ line attaches to the φ line in the loop via the new φ^3 vertex; however this diagram is finite. The story is the same for the loop correction to the φ^4 vertex; the divergent part of the fermion loop diagram is the same, and new diagrams with the φ^3 vertex are all finite. Thus the result is the same as eq. (51.53),

$$Z_\lambda = 1 + \left(\frac{3\lambda}{16\pi^2} - \frac{3g^4}{\pi^2\lambda}\right)\frac{1}{\varepsilon}\,. \tag{51.65}$$

Finally, we have to consider corrections to the new φ^3 vertex. There is a fermion loop diagram that yields

$$i\mathbf{V}_{3,\,\Psi\,\rm loop} = (-1)(ig)^3\left(\tfrac{1}{i}\right)^3\int\frac{d^4\ell}{(2\pi)^4}\,{\rm Tr}\,\tilde{S}(\not{\ell})\tilde{S}(\not{\ell}+\not{k}_1)\tilde{S}(\not{\ell}-\not{k}_2) + (k_2\leftrightarrow k_3)\,. \tag{51.66}$$

We can set the external momenta to zero; then the numerator becomes ${\rm Tr}\,(-\not{\ell}+m)^3 = 3m\,{\rm Tr}\,\not{\ell}^2 + m^3\,{\rm Tr}\,1 = -12m\ell^2 + 4m^3$, and only the ℓ^2 term contributes to the divergent part. The result is then $\mathbf{V}_{3,\Psi\,\rm loop} = 2(-1)(g^3)(-12m)/8\pi^2\varepsilon = 3g^3m/\pi^2\varepsilon$. There is also a φ-loop diagram, with one φ^3 vertex and one φ^4 vertex; there are three inequivalent permutations of the external momenta and a symmetry factor of $S=2$, and we have $\mathbf{V}_{3,\varphi\,\rm loop} = (\frac{1}{2})(3)(-i\lambda)(i\kappa)(1/i)^2(1/8\pi^2\varepsilon) = -3\lambda\kappa/16\pi^2\varepsilon$. Thus we find

$$Z_\kappa = 1 + \left(\frac{3\lambda}{16\pi^2} - \frac{3g^3m}{\pi^2\kappa}\right)\frac{1}{\varepsilon}\,. \tag{51.67}$$

52 Beta Functions in Yukawa Theory

52.1) We have

$$\begin{aligned}\gamma_\varphi &\equiv \frac{1}{2}\frac{d\ln Z_\varphi}{d\ln\mu} \\ &= \frac{1}{2}\left(-\tfrac{1}{2}g\frac{\partial}{\partial g} - \lambda\frac{\partial}{\partial\lambda}\right)\varepsilon\ln Z_\varphi \\ &= \frac{g^2}{8\pi^2}\,,\end{aligned} \tag{52.17}$$

$$\begin{aligned}\gamma_\Psi &\equiv \frac{1}{2}\frac{d\ln Z_\Psi}{d\ln\mu} \\ &= \frac{1}{2}\left(-\tfrac{1}{2}g\frac{\partial}{\partial g} - \lambda\frac{\partial}{\partial\lambda}\right)\varepsilon\ln Z_\Psi \\ &= \frac{g^2}{32\pi^2}\,,\end{aligned} \tag{52.18}$$

$$\begin{aligned}\gamma_m &\equiv \frac{d}{d\ln\mu}\ln m \\ &= \frac{d}{d\ln\mu}\Big[\ln m_0 - \ln\Big(Z_m/Z_\Psi\Big)\Big] \\ &= \left(\tfrac{1}{2}g\frac{\partial}{\partial g} + \lambda\frac{\partial}{\partial\lambda}\right)\varepsilon\ln\Big(Z_m/Z_\Psi\Big) \\ &= \frac{g^2}{16\pi^2}\,,\end{aligned} \tag{52.19}$$

$$\begin{aligned}\gamma_M &\equiv \frac{d}{d\ln\mu}\ln M \\ &= \frac{d}{d\ln\mu}\Big[\ln M_0 - \ln\Big(Z_M^{1/2}/Z_\varphi^{1/2}\Big)\Big] \\ &= \left(\tfrac{1}{2}g\frac{\partial}{\partial g} + \lambda\frac{\partial}{\partial\lambda}\right)\varepsilon\ln\Big(Z_M^{1/2}/Z_\varphi^{1/2}\Big) \\ &= \frac{g^2}{8\pi^2}\left(1 - 2\frac{m^2}{M^2}\right) + \frac{\lambda}{32\pi^2}\,.\end{aligned} \tag{52.20}$$

52.2) The values of Z_φ, Z_Ψ, Z_g, and Z_λ are the same in both theories, so the beta functions for g and λ and the anomalous dimensions of the fields are the same. To compute the beta function for κ, we note that $\kappa_0 = Z_\kappa Z_\varphi^{-3/2}\tilde\mu^{\varepsilon/2}\kappa$. If we let $\ln(Z_\kappa Z_\varphi^{-3/2}) = \sum_n K_n/\varepsilon^n$, then

$$\begin{aligned}\beta_\kappa &= \kappa\left(\tfrac{1}{2}g\frac{\partial}{\partial g} + \lambda\frac{\partial}{\partial\lambda} + \tfrac{1}{2}\kappa\frac{\partial}{\partial\kappa}\right)K_1 \\ &= \frac{1}{16\pi^2}\Big(6g^2\kappa + 3\lambda\kappa - 48g^3m\Big)\,.\end{aligned} \tag{52.21}$$

The values of Z_m and Z_M are different in the two theories, and we must also include the effect of the κ coupling; we have

$$\begin{aligned}\gamma_m &= \left(\tfrac{1}{2}g\frac{\partial}{\partial g} + \lambda\frac{\partial}{\partial\lambda} + \tfrac{1}{2}\kappa\frac{\partial}{\partial\kappa}\right)\varepsilon\ln\Big(Z_m/Z_\Psi\Big) \\ &= \frac{3g^2}{16\pi^2}\,, \end{aligned} \tag{52.22}$$

$$\begin{aligned}\gamma_M &= \left(\tfrac{1}{2}g\frac{\partial}{\partial g} + \lambda\frac{\partial}{\partial\lambda} + \tfrac{1}{2}\kappa\frac{\partial}{\partial\kappa}\right)\varepsilon\ln\Big(Z_M^{1/2}/Z_\varphi^{1/2}\Big) \\ &= \frac{g^2}{8\pi^2}\left(1-6\frac{m^2}{M^2}\right) + \frac{\kappa^2}{32\pi^2} + \frac{\lambda}{32\pi^2}\,. \end{aligned} \tag{52.23}$$

52.3) a&b) If $dg/d\ln\mu = b_0 g^3/16\pi^2$ and $d\lambda/d\ln\mu = (c_0 g^4 + c_1\lambda g^2 + c_2\lambda^2)/16\pi^2$, then for $\rho \equiv \lambda/g^2$ we have (by the chain rule)

$$\begin{aligned}\frac{d\rho}{d\ln\mu} &= \frac{g^2}{16\pi^2}\Big(c_0 + (c_1 - 2b_0)\rho + c_2\rho^2\Big) \\ &= \frac{g^2}{16\pi^2}c_2(\rho-\rho_+^*)(\rho-\rho_-^*)\,, \end{aligned} \tag{52.24}$$

where $\rho_\pm^* = [2b_0 - c_1 \pm \sqrt{(c_1-2b_0)^2 - 4c_0c_2}\,]/2c_2$. Eq. (52.24) is better because it is separable. For our case, $b_0 = 5$, $c_0 = -48$, $c_1 = 8$, $c_2 = 3$, and $\rho_\pm^* = (1\pm\sqrt{145})/3 = -3.68$ and $+4.32$.

c) Since g is small, we can treat it as approximately constant. For $\rho = 0$, β_ρ is positive, and so ρ increases as μ increases, and approaches ρ_+^* from below; ρ decreases as μ decreases, and approaches ρ_-^* from above.

d) Since the initial value of ρ is greater than ρ_+^*, β_ρ is negative, and ρ decreases as μ increases, approaching ρ_+^* from above; ρ increases as μ decreases, and grows without bound.

e) Since the initial value of ρ is less than ρ_-^*, β_ρ is negative, and ρ decreases as μ increases, growing more and more negative without bound; ρ increases as μ decreases, and approaches ρ_-^* from below.

f&g) We have $d\rho/dg = \beta_\rho/\beta_g = (c_2/b_0)(\rho-\rho_+^*)(\rho-\rho_-^*)/g^2$. This can be separated and integrated to get

$$\begin{aligned}\int\frac{d\rho}{(\rho-\rho_+^*)(\rho-\rho_-^*)} &= \frac{c_2}{b_0}\int\frac{dg}{g} \\ \frac{1}{\rho_+^*-\rho_-^*}\ln\left|\frac{\rho-\rho_+^*}{\rho-\rho_-^*}\right| &= \frac{c_2}{b_0}\ln|g/g_0|\,, \end{aligned} \tag{52.25}$$

which yields the claimed result with $\nu = b_0/[c_2(\rho_+^*-\rho_-^*)] = 5/2\sqrt{145} = 0.208$. Trajectories with $\rho < \rho_+^*$ flow towards $(\rho, g) = (\rho_-^*, \infty)$ as μ increases (towards the ultraviolet). Trajectories with $\rho > \rho_-^*$ flow towards $(\rho, g) = (\rho_+^*, 0)$ as μ decreases (towards the infrared). This explains the names.

53 Functional Determinants

54 Maxwell's Equations

55 Electrodynamics in Coulomb Gauge

55.1) The derivation is essentially the same as in problem 3.1 for a scalar field, with an extra three-vector index on the field and its conjugate momentum that is contracted with a polarization vector. Using eq. (55.13), we see that the $k_i k_j$ term in eq. (55.20) vanishes when contracted with the polarization vectors; using eq. (55.14), we see that the δ_{ij} term yields a factor of $\delta_{\lambda'\lambda}$.

55.2) Again, this mimics the scalar field case done in section 3. The only difference is that there is a product of polarization vectors that, using eq. (55.14), yields a factor of $\delta_{\lambda'\lambda}$ in the nonzero term.

56 LSZ Reduction for Photons

56.1) The derivation is the same as in problem 8.4, with the polarization vectors simply an additional factor.

57 The Path Integral for Photons

58 Spinor Electrodynamics

58.1) From section 40, with $j^\mu = e\overline{\Psi}\gamma^\mu\Psi$, we have

$$\begin{aligned} P^{-1}j^\mu(\mathbf{x},t)P &= +\mathcal{P}^\mu{}_\nu j^\nu(-\mathbf{x},t)\ , \\ T^{-1}j^\mu(\mathbf{x},t)T &= -\mathcal{T}^\mu{}_\nu j^\nu(\mathbf{x},-t)\ , \\ C^{-1}j^\mu(\mathbf{x},t)C &= -j^\nu(\mathbf{x},t)\ . \end{aligned} \tag{58.22}$$

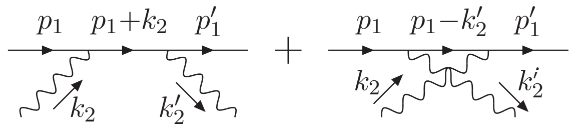
For $\mathcal{L}$ to be invariant, we then must have

$$\begin{aligned} P^{-1}A^\mu(\mathbf{x},t)P &= +\mathcal{P}^\mu{}_\nu A^\nu(-\mathbf{x},t)\ , \\ T^{-1}A^\mu(\mathbf{x},t)T &= -\mathcal{T}^\mu{}_\nu A^\nu(\mathbf{x},-t)\ , \\ C^{-1}A^\mu(\mathbf{x},t)C &= -A^\mu(\mathbf{x},t)\ . \end{aligned} \tag{58.23}$$

58.2) Such an amplitude would come from a correlation function $\langle 0|\mathrm{T}A^{\mu_1}(x_1)\ldots A^{\mu_n}(x_n)|0\rangle$ inserted into the LSZ formula. To see that this vanishes, we insert $1 = CC^{-1}$ between each pair of fields, and on the far left and far right. Since the vacuum is unique, it must be invariant under charge conjugation: $C^{-1}|0\rangle = |0\rangle$ and $\langle 0|C = \langle 0|$. Using $C^{-1}A^\mu C = -A^\mu$ from the previous problem, we see that this correlation function is equal to $(-1)^n$ times itself, and so must vanish if n is odd. Note that this means that the amplitude vanishes even if the photon momenta are off shell, so it also does not appear as a subdiagram in some other more complicated process.

59 Scattering in Spinor Electrodynamics

59.1) For $e^-\gamma \to e^-\gamma$, the diagrams are



and the amplitude is

$$\mathcal{T} = e^2 \varepsilon_2^{*\mu} \varepsilon_{2'}^{\nu}\, \overline{u}_1' A_{\mu\nu} u_1 \;, \tag{59.26}$$

where

$$A_{\mu\nu} \equiv \frac{\gamma_\nu(-\not p_1 - \not k_2 + m)\gamma_\mu}{-s+m^2} + \frac{\gamma_\mu(-\not p_1 + \not k_2' + m)\gamma_\nu}{-u+m^2} \;. \tag{59.27}$$

We have $\overline{A_{\mu\nu}} = A_{\nu\mu}$, and so

$$\overline{\mathcal{T}} = e^2 \varepsilon_2^{\rho} \varepsilon_{2'}^{*\sigma}\, \overline{u}_1 A_{\sigma\rho} u_1' \;. \tag{59.28}$$

Thus

$$|\mathcal{T}|^2 = e^4 (\varepsilon_2^{*\mu}\varepsilon_2^{\rho})(\varepsilon_{2'}^{\nu}\varepsilon_{2'}^{*\sigma}) \mathrm{Tr}[A_{\mu\nu}(u_1\overline{u}_1)A_{\sigma\rho}(u_1'\overline{u}_1')] \;. \tag{59.29}$$

Averaging over initial and summing over final spins and polarizations yields

$$\begin{aligned} \langle|\mathcal{T}|^2\rangle &= e^4\, \mathrm{Tr}[A_{\mu\nu}(-\not p_1 + m)A^{\nu\mu}(-\not p_1' + m)] \\ &= e^4 \Big[\frac{\langle\Phi_{ss}\rangle}{(m^2-s)^2} + \frac{\langle\Phi_{su}\rangle + \langle\Phi_{us}\rangle}{(m^2-s)(m^2-u)} + \frac{\langle\Phi_{uu}\rangle}{(m^2-u)^2} \Big] \,, \end{aligned} \tag{59.30}$$

where

$$\begin{aligned} \langle\Phi_{ss}\rangle &= \tfrac14 \mathrm{Tr}\Big[\gamma_\nu(-\not p_1 - \not k_2 + m)\gamma_\mu(-\not p_1 + m)\gamma^\mu(-\not p_1 - \not k_2 + m)\gamma^\nu(-\not p_1' + m)\Big] \,, \\ \langle\Phi_{uu}\rangle &= \tfrac14 \mathrm{Tr}\Big[\gamma_\mu(-\not p_1 + \not k_2' + m)\gamma_\nu(-\not p_1 + m)\gamma^\nu(-\not p_1 + \not k_2' + m)\gamma^\mu(-\not p_1' + m)\Big] \,, \\ \langle\Phi_{su}\rangle &= \tfrac14 \mathrm{Tr}\Big[\gamma_\nu(-\not p_1 - \not k_2 + m)\gamma_\mu(-\not p_1 + m)\gamma^\nu(-\not p_1 + \not k_2' + m)\gamma^\mu(-\not p_1' + m)\Big] \,, \\ \langle\Phi_{us}\rangle &= \tfrac14 \mathrm{Tr}\Big[\gamma_\mu(-\not p_1 + \not k_2' + m)\gamma_\nu(-\not p_1 + m)\gamma^\mu(-\not p_1 - \not k_2 + m)\gamma^\nu(-\not p_1' + m)\Big] \,. \end{aligned} \tag{59.31}$$

Examinging $\langle\Phi_{ss}\rangle$ and $\langle\Phi_{uu}\rangle$, we see that they are transformed into each other by $k_2 \leftrightarrow -k_2'$, which is equivalent to $s \leftrightarrow u$. The same is true of $\langle\Phi_{su}\rangle$ and $\langle\Phi_{us}\rangle$. Thus we need only compute $\langle\Phi_{ss}\rangle$ and $\langle\Phi_{su}\rangle$, and then take $s \leftrightarrow u$ to get $\langle\Phi_{uu}\rangle$ and $\langle\Phi_{us}\rangle$. Using $\gamma^\mu\gamma_\mu = -4$, $\gamma^\mu \not p \gamma_\mu = 2\not p$, and $\mathrm{Tr}[\not p \not q] = -4pq$, we have

$$\begin{aligned} \langle\Phi_{ss}\rangle &= \tfrac14 \mathrm{Tr}[(-\not p_1 - \not k_2 + m)(-2\not p_1 - 4m)(-\not p_1 - \not k_2 + m)(-2\not p_1' - 4m)] \\ &= \mathrm{Tr}[(\not p_1 + \not k_2)\not p_1(\not p_1 + \not k_2)\not p_1'] \\ &\quad + m^2\, \mathrm{Tr}[4(\not p_1 + \not k_2)(\not p_1 + \not k_2) - 4\not p_1(\not p_1 + \not k_2) - 4\not p_1'(\not p_1 + \not k_2) + \not p_1 \not p_1'] \\ &\quad + 4m^4\, \mathrm{Tr}\, 1 \\ &= 8[p_1(p_1{+}k_2)][p_1'(p_1{+}k_2)] - 4(p_1{+}k_2)^2 p_1p_1' \\ &\quad - 16m^2(p_1{+}k_2)^2 + 16m^2 p_1(p_1{+}k_2) + 16m^2 p_1'(p_1{+}k_2) - 4m^2 p_1 p_1' \\ &\quad + 16m^4 \,. \end{aligned} \tag{59.32}$$

Now we use

$$\begin{aligned}
(p_1{+}k_2)^2 &= -s\ ,\\
p_1p_1' &= -\tfrac12(t-2m^2) = -\tfrac12(s+u)\ ,\\
p_1(p_1{+}k_2) &= -\tfrac12(s+m^2)\ ,\\
p_1'(p_1{+}k_2) &= -\tfrac12(s+m^2)\ .
\end{aligned}\tag{59.33}$$

The last equality follows from $p_1+k_2=p_1'+k_2'$ and $s=-(p_1'+k_2')^2$. For later use we note also that

$$\begin{aligned}
p_1(p_1{-}k_2') &= -\tfrac12(u+m^2)\ ,\\
p_1'(p_1{-}k_2') &= -\tfrac12(u+m^2)\ ,\\
(p_1{+}k_2)(p_1{-}k_2') &= -m^2\ .
\end{aligned}\tag{59.34}$$

We then have

$$\langle\Phi_{ss}\rangle = 2(s+m^2)^2 - 2s(s+u) + 16m^2 s - 16m^2(s+m^2) + 2m^2(s+u) + 16m^4\ . \tag{59.35}$$

This simplifies to

$$\langle\Phi_{ss}\rangle = -2[su - m^2(3s+u) - m^4]\ , \tag{59.36}$$

and swapping s and u yields

$$\langle\Phi_{uu}\rangle = -2[su - m^2(3u+s) - m^4]\ . \tag{59.37}$$

For $\langle\Phi_{su}\rangle$, we have

$$\begin{aligned}
\langle\Phi_{su}\rangle = {} & \tfrac14\mathrm{Tr}[\gamma_\nu(\not p_1{+}\not k_2)\gamma_\mu\not p_1\gamma^\nu(\not p_1{-}\not k_2')\gamma^\mu\not p_1']\\
&+\tfrac14 m^2\,\mathrm{Tr}[\gamma_\nu(\not p_1{+}\not k_2)\gamma_\mu\not p_1\gamma^\nu\gamma^\mu]\\
&+\tfrac14 m^2\,\mathrm{Tr}[\gamma_\nu(\not p_1{+}\not k_2)\gamma_\mu\gamma^\nu(\not p_1{-}\not k_2')\gamma^\mu]\\
&+\tfrac14 m^2\,\mathrm{Tr}[\gamma_\nu(\not p_1{+}\not k_2)\gamma_\mu\gamma^\nu\gamma^\mu\not p_1']\\
&+\tfrac14 m^2\,\mathrm{Tr}[\gamma_\nu\gamma_\mu\not p_1\gamma^\nu(\not p_1{-}\not k_2')\gamma^\mu]\\
&+\tfrac14 m^2\,\mathrm{Tr}[\gamma_\nu\gamma_\mu\not p_1\gamma^\nu\gamma^\mu\not p_1']\\
&+\tfrac14 m^2\,\mathrm{Tr}[\gamma_\nu\gamma_\mu\gamma^\nu(\not p_1{-}\not k_2')\gamma^\mu\not p_1']\\
&+\tfrac14 m^4\,\mathrm{Tr}[\gamma_\nu\gamma_\mu\gamma^\nu\gamma^\mu]\ .
\end{aligned}\tag{59.38}$$

We use $\gamma_\nu\not p\gamma^\nu = 2\not p$, $\gamma_\nu\not p\not q\gamma^\nu = 4pq$, $\gamma_\nu\not p\not q\not r\gamma^\nu = 2\not r\not q\not p$, and $\gamma_\mu\gamma^\mu=-4$ to get

$$\begin{aligned}
\langle\Phi_{su}\rangle = {} & \tfrac12\mathrm{Tr}[\not p_1\gamma_\mu(\not p_1{+}\not k_2)(\not p_1{-}\not k_2')\gamma^\mu\not p_1']\\
&+\tfrac12 m^2\,\mathrm{Tr}[(\not p_1{+}\not k_2)\gamma_\mu\not p_1\gamma^\mu]\\
&+m^2(p_1{+}k_2)_\mu\mathrm{Tr}[(\not p_1{-}\not k_2')\gamma^\mu]\\
&+\tfrac12 m^2\,\mathrm{Tr}[\gamma_\nu(\not p_1{+}\not k_2)\gamma^\nu\not p_1']\\
&+\tfrac12 m^2\,\mathrm{Tr}[\gamma_\nu\not p_1\gamma^\nu(\not p_1{-}\not k_2')]\\
&+m^2\,p_{1\mu}\mathrm{Tr}[\gamma^\mu\not p_1']\\
&+\tfrac12 m^2\,\mathrm{Tr}[\gamma_\mu(\not p_1{-}\not k_2')\gamma^\mu\not p_1']\\
&-2m^4\,\mathrm{Tr}\,1\ .
\end{aligned}\tag{59.39}$$

Now we use $\gamma_\nu \not{p} \gamma^\nu = 2\not{p}$ and $\gamma_\nu \not{p}\not{q} \gamma^\nu = 4pq$ to get

$$\begin{aligned}\langle\Phi_{su}\rangle = {} & 2(p_1{+}k_2)(p_1{-}k_2')\mathrm{Tr}[\not{p}_1\not{p}_1'] \\ & + m^2\,\mathrm{Tr}[(\not{p}_1{+}\not{k}_2)\not{p}_1] \\ & + m^2\,\mathrm{Tr}[(\not{p}_1{-}\not{k}_2')(\not{p}_1{+}\not{k}_2)] \\ & + m^2\,\mathrm{Tr}[(\not{p}_1{+}\not{k}_2)\not{p}_1'] \\ & + m^2\,\mathrm{Tr}[\not{p}_1(\not{p}_1{-}\not{k}_2')] \\ & + m^2\,\mathrm{Tr}[\not{p}_1\not{p}_1'] \\ & + m^2\,\mathrm{Tr}[(\not{p}_1{-}\not{k}_2')\not{p}_1'] \\ & - 2m^4\,\mathrm{Tr}\,1\,. \end{aligned} \tag{59.40}$$

Taking the traces and using eqs. (59.33) and (59.34), we get

$$\langle\Phi_{su}\rangle = 4m^2[-(s{+}u) + \tfrac{1}{2}(s{+}m^2) + m^2 + \tfrac{1}{2}(s{+}m^2) + \tfrac{1}{2}(u{+}m^2) + \tfrac{1}{2}(s{+}u) + \tfrac{1}{2}(u{+}m^2) - 2m^2]. \tag{59.41}$$

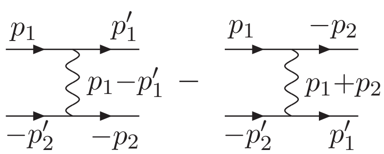
Using $s + t + u = 2m^2$, this simplifies to

$$\langle\Phi_{ss}\rangle = -2m^2(t - 4m^2)\;, \tag{59.42}$$

and swapping s and u yields

$$\langle\Phi_{uu}\rangle = -2m^2(t - 4m^2)\;. \tag{59.43}$$

59.2) For $e^+e^- \to e^+e^-$, the diagrams are



and the amplitude is

$$\mathcal{T} = e^2\left[\frac{(\overline{u}_1'\gamma^\mu u_1)(\overline{v}_2\gamma_\mu v_2')}{-t} - \frac{(\overline{v}_2\gamma^\mu u_1)(\overline{u}_1'\gamma_\mu v_2')}{-s}\right]. \tag{59.44}$$

We then have

$$\overline{\mathcal{T}} = e^2\left[\frac{(\overline{u}_1\gamma^\nu u_1')(\overline{v}_2'\gamma_\nu v_2)}{-t} - \frac{(\overline{u}_1\gamma^\nu v_2)(\overline{v}_2'\gamma_\nu u_1')}{-s}\right]. \tag{59.45}$$

Thus

$$|\mathcal{T}|^2 = e^4\Big[\Phi_{tt}/t^2 + (\Phi_{ts}+\Phi_{st})/ts + \Phi_{ss}/s^2\Big]\;, \tag{59.46}$$

where

$$\begin{aligned}\Phi_{tt} &= \mathrm{Tr}[(u_1\overline{u}_1)\gamma^\nu(u_1'\overline{u}_1')\gamma^\mu]\,\mathrm{Tr}[(v_2\overline{v}_2)\gamma_\mu(v_2'\overline{v}_2')\gamma_\nu]\;, \\ \Phi_{ss} &= \mathrm{Tr}[(u_1\overline{u}_1)\gamma^\nu(v_2\overline{v}_2)\gamma^\mu]\,\mathrm{Tr}[(u_1'\overline{u}_1')\gamma_\mu(v_2'\overline{v}_2')\gamma_\nu]\;, \\ \Phi_{ts} &= \mathrm{Tr}[(u_1\overline{u}_1)\gamma^\nu(v_2\overline{v}_2)\gamma_\mu(v_2'\overline{v}_2')\gamma_\nu(u_1'\overline{u}_1')\gamma^\mu]\;, \\ \Phi_{st} &= \mathrm{Tr}[(u_1\overline{u}_1)\gamma^\nu(u_1'\overline{u}_1')\gamma_\mu(v_2'\overline{v}_2')\gamma_\nu(v_2\overline{v}_2)\gamma^\mu]\;. \end{aligned} \tag{59.47}$$

Averaging over initial spins and summing over final spins yields

$$\begin{aligned}
\langle\Phi_{tt}\rangle &= \tfrac{1}{4}\mathrm{Tr}[(-\not{p}_1{+}m)\gamma^\nu(-\not{p}_1'{+}m)\gamma^\mu]\,\mathrm{Tr}[(-\not{p}_2{-}m)\gamma_\mu(-\not{p}_2'{-}m)\gamma_\nu]\;,\\
\langle\Phi_{ss}\rangle &= \tfrac{1}{4}\mathrm{Tr}[(-\not{p}_1{+}m)\gamma^\nu(-\not{p}_2{-}m)\gamma^\mu]\,\mathrm{Tr}[(-\not{p}_1'{+}m)\gamma_\mu(-\not{p}_2'{-}m)\gamma_\nu]\;,\\
\langle\Phi_{ts}\rangle &= \tfrac{1}{4}\mathrm{Tr}[(-\not{p}_1{+}m)\gamma^\nu(-\not{p}_2{-}m)\gamma_\mu(-\not{p}_2'{-}m)\gamma_\nu(-\not{p}_1'{+}m)\gamma^\mu]\;,\\
\langle\Phi_{st}\rangle &= \tfrac{1}{4}\mathrm{Tr}[(-\not{p}_1{+}m)\gamma^\nu(-\not{p}_1'{+}m)\gamma_\mu(-\not{p}_2'{-}m)\gamma_\nu(-\not{p}_2{-}m)\gamma^\mu]\;.
\end{aligned} \tag{59.48}$$

We see that exchanging $p_1' \leftrightarrow -p_2$, which is equivalent to $t \leftrightarrow s$, exchanges $\langle\Phi_{tt}\rangle \leftrightarrow \langle\Phi_{ss}\rangle$ and $\langle\Phi_{ts}\rangle \leftrightarrow \langle\Phi_{st}\rangle$. We have

$$\begin{aligned}
\langle\Phi_{tt}\rangle &= \tfrac{1}{4}\Big(\mathrm{Tr}[\not{p}_1\gamma^\nu\not{p}_1'\gamma^\mu] + m^2\,\mathrm{Tr}[\gamma^\nu\gamma^\mu]\Big)\Big(\mathrm{Tr}[\not{p}_2\gamma_\mu\not{p}_2'\gamma_\nu] + m^2\,\mathrm{Tr}[\gamma_\mu\gamma_\nu]\Big)\\
&= 4\Big(p_1^\nu p_1'^{\,\mu} + p_1^\mu p_1'^{\,\nu} - (p_1p_1' + m^2)g^{\mu\nu}\Big)\Big(p_{2\mu}p_{2\nu}' + p_{2\nu}p_{2\mu}' - (p_2p_2' + m^2)g_{\nu\mu}\Big)\\
&= 4[2(p_1p_2')(p_1'p_2) + 2(p_1p_2)(p_1'p_2') - 2(p_1p_1')(p_2p_2' + m^2) - 2(p_2p_2')(p_1p_1' + m^2)\\
&\qquad + 4(p_1p_1' + m^2)(p_2p_2' + m^2)]\;.
\end{aligned} \tag{59.49}$$

Now we use

$$\begin{aligned}
p_1p_2 = p_1'p_2' &= -\tfrac{1}{2}(s - 2m^2)\;,\\
p_1p_1' = p_2p_2' &= +\tfrac{1}{2}(t - 2m^2)\;,\\
p_1p_2' = p_1'p_2 &= +\tfrac{1}{2}(u - 2m^2) = \tfrac{1}{2}(2m^2 - s - t)
\end{aligned} \tag{59.50}$$

and simplify to get

$$\langle\Phi_{tt}\rangle = 2(t^2 + 2st + 2s^2 - 8m^2s + 8m^4)\;. \tag{59.51}$$

Swapping $t \leftrightarrow s$ yields

$$\langle\Phi_{ss}\rangle = 2(s^2 + 2st + 2t^2 - 8m^2t + 8m^4)\;. \tag{59.52}$$

Next we have

$$\begin{aligned}
\langle\Phi_{ts}\rangle = \;&\tfrac{1}{4}\mathrm{Tr}[\not{p}_1\gamma^\nu\not{p}_2\gamma_\mu\not{p}_2'\gamma_\nu\not{p}_1'\gamma^\mu]\\
&- \tfrac{1}{4}m^2\,\mathrm{Tr}[\not{p}_1\gamma^\nu\not{p}_2\gamma_\mu\gamma_\nu\gamma^\mu]\\
&- \tfrac{1}{4}m^2\,\mathrm{Tr}[\not{p}_1\gamma^\nu\gamma_\mu\not{p}_2'\gamma_\nu\gamma^\mu]\\
&+ \tfrac{1}{4}m^2\,\mathrm{Tr}[\not{p}_1\gamma^\nu\gamma_\mu\gamma_\nu\not{p}_1'\gamma^\mu]\\
&+ \tfrac{1}{4}m^2\,\mathrm{Tr}[\gamma^\nu\not{p}_2\gamma_\mu\not{p}_2'\gamma_\nu\gamma^\mu]\\
&- \tfrac{1}{4}m^2\,\mathrm{Tr}[\gamma^\nu\not{p}_2\gamma_\mu\gamma_\nu\not{p}_1'\gamma^\mu]\\
&- \tfrac{1}{4}m^2\,\mathrm{Tr}[\gamma^\nu\gamma_\mu\not{p}_2'\gamma_\nu\not{p}_1'\gamma^\mu]\\
&+ \tfrac{1}{4}m^4\,\mathrm{Tr}[\gamma^\nu\gamma_\mu\gamma_\nu\gamma^\mu]\;.
\end{aligned} \tag{59.53}$$

In the first line, $\not{p}_1\gamma^\nu\not{p}_2\gamma_\mu\not{p}_2'\gamma_\nu\not{p}_1'\gamma^\mu = 2\not{p}_1\not{p}_2'\gamma_\mu\not{p}_2\not{p}_1'\gamma^\mu = (2\not{p}_1\not{p}_2')(4p_2p_1')$. In the second line, $\not{p}_1\gamma^\nu\not{p}_2\gamma_\mu\gamma_\nu\gamma^\mu = 4\not{p}_1p_{2\mu}\gamma^\mu = 4\not{p}_1\not{p}_2$; the next five lines can be similarly simplified. In the last line, $\gamma^\nu\gamma_\mu\gamma_\nu\gamma^\mu = 2\gamma_\mu\gamma^\mu = -8$. Taking traces yields

$$\langle\Phi_{ts}\rangle = -8(p_1p_2')(p_1'p_2) + 4m^2(p_1p_2 + p_1p_2' - p_1p_1' - p_2p_2' + p_1'p_2 + p_1'p_2') - 8m^4\;, \tag{59.54}$$

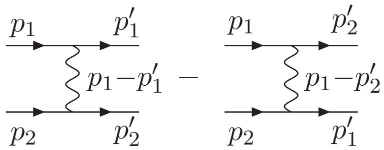
and plugging in eq. (59.50), we find

$$\langle \Phi_{ts} \rangle = -2(u^2 - 8m^2 u + 12m^4) \; . \tag{59.55}$$

Swapping $t \leftrightarrow s$, we get

$$\langle \Phi_{st} \rangle = -2(u^2 - 8m^2 u + 12m^4) \; . \tag{59.56}$$

59.3) For $e^- e^- \to e^- e^-$, the diagrams are



and the amplitude is

$$\mathcal{T} = e^2 \left[\frac{(\overline{u}'_1 \gamma^\mu u_1)(\overline{u}'_2 \gamma_\mu u_2)}{-t} - \frac{(\overline{u}'_2 \gamma^\mu u_1)(\overline{u}'_1 \gamma_\mu u_2)}{-u} \right] . \tag{59.57}$$

We then have

$$\overline{\mathcal{T}} = e^2 \left[\frac{(\overline{u}_1 \gamma^\nu u'_1)(\overline{u}_2 \gamma_\nu u'_2)}{-t} - \frac{(\overline{u}_1 \gamma^\nu u'_2)(\overline{u}_2 \gamma_\nu u'_1)}{-s} \right] . \tag{59.58}$$

Thus

$$|\mathcal{T}|^2 = e^4 \Big[\Phi_{tt}/t^2 + (\Phi_{tu} + \Phi_{ut})/tu + \Phi_{uu}/u^2 \Big] \; , \tag{59.59}$$

where

$$\begin{aligned}
\Phi_{tt} &= \mathrm{Tr}[(u_1 \overline{u}_1)\gamma^\nu (u'_1 \overline{u}'_1)\gamma^\mu] \, \mathrm{Tr}[(u_2 \overline{u}_2)\gamma_\nu (u'_2 \overline{u}'_2)\gamma_\mu] \; , \\
\Phi_{uu} &= \mathrm{Tr}[(u_1 \overline{u}_1)\gamma^\nu (u'_2 \overline{u}'_2)\gamma^\mu] \, \mathrm{Tr}[(u'_1 \overline{u}'_1)\gamma_\mu (u_2 \overline{u}_2)\gamma_\nu] \; , \\
\Phi_{tu} &= \mathrm{Tr}[(u_1 \overline{u}_1)\gamma^\nu (u'_2 \overline{u}'_2)\gamma_\mu (u_2 \overline{u}_2)\gamma_\nu (u'_1 \overline{u}'_1)\gamma^\mu] \; , \\
\Phi_{ut} &= \mathrm{Tr}[(u_1 \overline{u}_1)\gamma^\nu (u'_1 \overline{u}'_1)\gamma_\mu (u_2 \overline{u}_2)\gamma_\nu (u'_2 \overline{u}'_2)\gamma^\mu] \; .
\end{aligned} \tag{59.60}$$

Averaging over initial spins and summing over final spins yields

$$\begin{aligned}
\langle \Phi_{tt} \rangle &= \tfrac{1}{4} \mathrm{Tr}[(-\not{p}_1 + m)\gamma^\nu (-\not{p}'_1 + m)\gamma^\mu] \, \mathrm{Tr}[(-\not{p}_2 + m)\gamma_\mu (-\not{p}'_2 + m)\gamma_\nu] \; , \\
\langle \Phi_{uu} \rangle &= \tfrac{1}{4} \mathrm{Tr}[(-\not{p}_1 + m)\gamma^\nu (-\not{p}'_2 + m)\gamma^\mu] \, \mathrm{Tr}[(-\not{p}'_1 + m)\gamma_\mu (-\not{p}_2 + m)\gamma_\nu] \; , \\
\langle \Phi_{tu} \rangle &= \tfrac{1}{4} \mathrm{Tr}[(-\not{p}_1 + m)\gamma^\nu (-\not{p}'_2 + m)\gamma_\mu (-\not{p}_2 + m)\gamma_\nu (-\not{p}'_1 + m)\gamma^\mu] \; , \\
\langle \Phi_{ut} \rangle &= \tfrac{1}{4} \mathrm{Tr}[(-\not{p}_1 + m)\gamma^\nu (-\not{p}'_1 + m)\gamma_\mu (-\not{p}_2 + m)\gamma_\nu (-\not{p}'_2 + m)\gamma^\mu] \; .
\end{aligned} \tag{59.61}$$

Comparing with eq. (59.48) for $e^+ e^- \to e^+ e^-$, we see that eq. (59.61) is transformed into eq. (59.61) via $p_2 \leftrightarrow -p'_2$. This is equivalent to $s \leftrightarrow u$, so we need not redo the calculation; we have

$$\begin{aligned}
\langle \Phi_{tt} \rangle &= 2(t^2 + 2tu + 2u^2 - 8m^2 u + 8m^4) \; , \\
\langle \Phi_{uu} \rangle &= 2(u^2 + 2tu + 2t^2 - 8m^2 t + 8m^4) \; , \\
\langle \Phi_{tu} \rangle &= -2(s^2 - 8m^2 s + 12m^4) \; , \\
\langle \Phi_{ut} \rangle &= -2(s^2 - 8m^2 s + 12m^4) \; .
\end{aligned} \tag{59.62}$$

60 Spinor Helicity for Spinor Electrodynamics

60.1) a) Using eqs. (60.7–8) and $-\not{p} = |p\rangle[p| + |p]\langle p|$, we have

$$p_\mu \varepsilon_+^\mu(k) = -\frac{\langle q|\not{p}|k]}{\sqrt{2}\,\langle q\,k\rangle} = \frac{\langle q\,p\rangle\,[p\,k]}{\sqrt{2}\,\langle q\,k\rangle}\,, \tag{60.43}$$

$$p_\mu \varepsilon_-^\mu(k) = -\frac{[q|\not{p}|k\rangle}{\sqrt{2}\,[q\,k]} = \frac{[q\,p]\,\langle p\,k\rangle}{\sqrt{2}\,[q\,k]}\,. \tag{60.44}$$

Setting $p = q$ or $p = k$ makes both expressions vanish, since $\langle q\,q\rangle = [q\,q] = 0$.

b) We need $\langle q|\gamma^\mu|k] = [k|\gamma^\mu|q\rangle$ and the Fierz identity $[p|\gamma^\mu|q\rangle\,\langle r|\gamma_\mu|s] = 2\,[p\,s]\,\langle q\,r\rangle$, both proved in problem 50.5. Then using eqs. (60.7–8), we have

$$\varepsilon_+(k;q)\cdot\varepsilon_+(k';q') = \frac{[k|\gamma^\mu|q\rangle\,\langle q'|\gamma^\mu|k']}{2\,\langle q\,k\rangle\,\langle q'\,k'\rangle} = \frac{\langle q\,q'\rangle\,[k\,k']}{\langle q\,k\rangle\,\langle q'\,k'\rangle}\,, \tag{60.45}$$

$$\varepsilon_-(k;q)\cdot\varepsilon_-(k';q') = \frac{[q|\gamma^\mu|k\rangle\,\langle k'|\gamma^\mu|q']}{2\,[q\,k]\,[q'\,k']} = \frac{[q\,q']\,\langle k\,k'\rangle}{[q\,k]\,[q'\,k']}\,, \tag{60.46}$$

$$\varepsilon_+(k;q)\cdot\varepsilon_-(k';q') = \frac{[k|\gamma^\mu|q\rangle\,\langle k'|\gamma^\mu|q']}{2\,\langle q\,k\rangle\,[q'\,k']} = \frac{\langle q\,k'\rangle\,[k\,q']}{\langle q\,k\rangle\,[q'\,k']}\,. \tag{60.47}$$

60.2) a) If p_j is the four-momentum particle j (with the convention that all momenta are outgoing), then $\sum_j p_j = 0$, and so $\sum_j \not{p}_j = -\sum_j(|j\rangle[j| + |j]\langle j|) = 0$. Sandwiching this between $\langle i|$ and $|k]$ yields $\sum_j \langle i\,j\rangle\,[j\,k] = 0$.

b) Since $\langle i\,i\rangle = [i\,i] = 0$, $j = i$ and $j = k$ do not contribute to the sum. For $n = 4$, this yields $\langle 2\,1\rangle\,[1\,3] + \langle 2\,4\rangle\,[4\,3] = 0$, or equivalently $[3\,1]\,\langle 1\,2\rangle = -\,[3\,4]\,\langle 4\,2\rangle$.

60.3) Multiply the numerator and denominator of $\langle 2\,4\rangle^2/\langle 1\,3\rangle\langle 2\,3\rangle$ by $[2\,4]^2$. In the numerator, use $\langle 2\,4\rangle^2\,[2\,4]^2 = s_{24}^2 = s_{13}^2 = \langle 1\,3\rangle^2\,[1\,3]^2$. In the denominator, use $\langle 2\,3\rangle\,[2\,4] = -\langle 1\,3\rangle\,[1\,4]$, which follows from momentum conservation. The result is $-[1\,3]^2/[1\,4]\,[2\,4]$.

60.4) We have

$$\mathcal{T}_{+-+-} = -2e^2\,\frac{\langle 2\,4\rangle\,[q_4|(\not{p}_1 + \not{k}_3)|2\rangle\,[3\,1]}{[q_4 4]\,\langle 2\,3\rangle\,s_{13}}\,. \tag{60.48}$$

Using $-\not{p} = |p\rangle[p| + |p]\langle p|$ in the numerator, and $s_{13} = \langle 1\,3\rangle\,[3\,1]$ in the denominator, we get

$$\mathcal{T}_{+-+-} = 2e^2\,\frac{\langle 2\,4\rangle\,([q_4 1]\,\langle 1\,2\rangle + [q_4 3]\,\langle 3\,2\rangle)}{[q_4 4]\,\langle 2\,3\rangle\,\langle 1\,3\rangle}\,. \tag{60.49}$$

a) We take $q_4 = p_1$, and so

$$\mathcal{T}_{+-+-} = 2e^2\,\frac{\langle 2\,4\rangle\,[1\,3]\,\langle 3\,2\rangle}{[1\,4]\,\langle 2\,3\rangle\,\langle 1\,3\rangle}\,. \tag{60.50}$$

In the numerator, use $[1\,3]\,\langle 3\,2\rangle = -[1\,4]\,\langle 4\,2\rangle = [1\,4]\,\langle 2\,4\rangle$ and cancel common factors to get eq (60.31).

b) We take $q_4 = p_2$, and so

$$\begin{aligned}
\mathcal{T}_{+-+-} &= 2e^2 \, \frac{\langle 2\,4\rangle\,([2\,1]\,\langle 1\,2\rangle + [2\,3]\,\langle 3\,2\rangle)}{[2\,4]\,\langle 2\,3\rangle\,\langle 1\,3\rangle} \\
&= 2e^2 \, \frac{\langle 2\,4\rangle\,(s_{12} + s_{23})}{[2\,4]\,\langle 2\,3\rangle\,\langle 1\,3\rangle} \\
&= 2e^2 \, \frac{\langle 2\,4\rangle\,(-s_{24})}{[2\,4]\,\langle 2\,3\rangle\,\langle 1\,3\rangle} \\
&= 2e^2 \, \frac{\langle 2\,4\rangle\,(\langle 2\,4\rangle\,[2\,4])}{[2\,4]\,\langle 2\,3\rangle\,\langle 1\,3\rangle} \\
&= 2e^2 \, \frac{\langle 2\,4\rangle^2}{\langle 1\,3\rangle\langle 2\,3\rangle} \; . \qquad (60.51)
\end{aligned}$$

61 Scalar Electrodynamics

61.1) We have

$$\mathcal{T} = -4e^2 \varepsilon_{1'}^{\mu} \varepsilon_{2'}^{\nu} \left[\frac{k_1^\mu k_2^\nu}{m^2 - t} + \frac{k_1^\nu k_2^\mu}{m^2 - u} + \tfrac{1}{2} g^{\mu\nu} \right], \tag{61.16}$$

and so

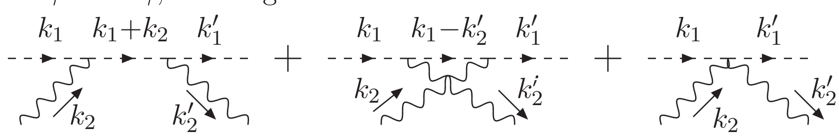
$$\overline{\mathcal{T}} = -4e^2 \varepsilon_{1'}^{*\rho} \varepsilon_{2'}^{*\sigma} \left[\frac{k_1^\rho k_2^\sigma}{m^2 - t} + \frac{k_1^\sigma k_2^\rho}{m^2 - u} + \tfrac{1}{2} g^{\mu\nu} \right]. \tag{61.17}$$

After summing over final polarizations, we have

$$\begin{aligned} \langle |\mathcal{T}|^2 \rangle &= 16e^4 \left[\frac{k_1^\mu k_2^\nu}{m^2 - t} + \frac{k_1^\nu k_2^\mu}{m^2 - u} + \tfrac{1}{2} g^{\mu\nu} \right] \left[\frac{k_{1\mu} k_{2\nu}}{m^2 - t} + \frac{k_{1\nu} k_{2\mu}}{m^2 - u} + \tfrac{1}{2} g_{\mu\nu} \right] \\ &= 16e^4 \left[\frac{m^4}{(m^2 - t)^2} + \frac{m^4}{(m^2 - u)^2} + 1 + \frac{2(k_1 k_2)^2}{(m^2 - t)(m^2 - u)} + \frac{k_1 k_2}{m^2 - t} + \frac{k_1 k_2}{m^2 - u} \right], \end{aligned} \tag{61.18}$$

where $k_1 k_2 = -\frac{1}{2}(s - 2m^2) = \frac{1}{2}(t + u)$.

61.2) For $\widetilde{e}^- \gamma \to \widetilde{e}^- \gamma$, the diagrams are



and the amplitude is

$$\mathcal{T} = e^2 \varepsilon_2^{*\mu} \varepsilon_{2'}^{\nu} \left[\frac{(2k_1 + k_2)_\mu (k_1 + k_1' + k_2)_\nu}{m^2 - s} + \frac{(k_1 - k_2' + k_1')_\mu (2k_1 - k_2')_\nu}{m^2 - u} - 2g_{\mu\nu} \right]. \tag{61.19}$$

We use $k_1 + k_2 = k_1' + k_2'$ to replace $k_1 + k_1' + k_2$ with $2k_1' + k_2'$ and $k_1 - k_2' + k_1'$ with $2k_1' - k_2$, and then use $k_2 \cdot \varepsilon_2^* = k_2' \cdot \varepsilon_{2'} = 0$ to get

$$\mathcal{T} = e^2 \varepsilon_2^{*\mu} \varepsilon_{2'}^{\nu} \left[\frac{4k_{1\mu} k_{1\nu}'}{m^2 - s} + \frac{4k_{1\mu}' k_{1\nu}}{m^2 - u} - 2g_{\mu\nu} \right]. \tag{61.20}$$

Squaring, averaging over the initial polarization, and summing over the final polarization yields

$$\begin{aligned} \langle |\mathcal{T}|^2 \rangle &= 8e^4 \left[\frac{k_1^\mu k_1'^{\,\nu}}{m^2 - s} + \frac{k_1'^{\,\mu} k_1^\nu}{m^2 - u} - \tfrac{1}{2} g^{\mu\nu} \right] \left[\frac{k_{1\mu} k_{1\nu}'}{m^2 - s} + \frac{k_{1\mu}' k_{1\nu}}{m^2 - u} - \tfrac{1}{2} g_{\mu\nu} \right] \\ &= 8e^4 \left[\frac{m^4}{(m^2 - s)^2} + \frac{m^4}{(m^2 - u)^2} + 1 + \frac{2(k_1 k_1')^2}{(m^2 - s)(m^2 - u)} - \frac{k_1 k_1'}{m^2 - s} - \frac{k_1 k_1'}{m^2 - u} \right], \end{aligned} \tag{61.21}$$

where $k_1 k_1' = \frac{1}{2}(t - 2m^2) = -\frac{1}{2}(s + u)$. This is related by $s \leftrightarrow t$ to eq. (61.18). There is an extra factor of 2 in eq. (61.18) because both polarizations are summed in that case (instead of one being averaged).

62 Loop Corrections in Spinor Electrodynamics

62.1) In momentum space, the gauge-fixing term becomes $-\frac{1}{2}\xi^{-1}k^\mu k^\nu \widetilde{A}_\mu(k)\widetilde{A}_\nu(-k)$. Adding this to eq. (57.3) yields $-\frac{1}{2}\widetilde{A}_\mu(k)[k^2 P^{\mu\nu}(k) + \xi^{-1}k^\mu k^\nu]\widetilde{A}_\nu(-k)$ as the kinetic term for the gauge field. The propagator is the matrix inverse of the contents of the square brackets. Since $P^{\mu\nu}(k)$ and $k^\mu k^\nu/k^2$ are orthogonal projection matrices, the propagator is $(1/k^2)[P^{\mu\nu}(k) + \xi k^\mu k^\nu/k^2]$. In the limit $\xi \to 0$, including this term in the lagrangian yields a path integrand that oscillates infinitely rapidly whenever $\partial^\mu A_\mu \neq 0$; thus the path integral vanishes unless $\partial^\mu A_\mu = 0$, and sothe $\xi \to 0$ limit corresponds to Lorenz gauge.

62.2) Only the photon propagator is changed. Since the one-loop contribution to $\Pi^{\mu\nu}(k)$ does not include a photon propagator, Z_3 is unchanged at one loop. The extra term $(\xi{-}1)k^\mu k^\nu/(k^2)^2$ in the photon propagator would add an extra term to the electron self-energy of the form

$$i\Delta\Sigma(p) = (\xi{-}1)e^2 \int \frac{d^4\ell}{(2\pi)^4} \frac{\not\ell(-\not p - \not\ell + m)\not\ell}{((p{+}\ell)^2 + m^2)(\ell^2)^2} - i(\Delta Z_2)\not p - i(\Delta Z_m)m \; , \tag{62.51}$$

where ΔZ_2 and ΔZ_m are the extra contributions to Z_2 and Z_m that are needed to cancel the extra contributions to the divergence. Combining denominators with Feynman's formula yields

$$\begin{aligned} \frac{1}{((p{+}l)^2 + m^2)(l^2)^2} &= \int dF_3\, [x_1(p{+}\ell)^2 + x_1 m^2 + x_2\ell^2 + x_3\ell^2]^{-3} \\ &= \int dF_3\, [\ell^2 + 2x_1\ell{\cdot}p + x_1p^2 + x_1m^2]^{-3} \\ &= \int dF_3\, [(\ell{+}x_1p)^2 + x_1(1{-}x_1)p^2 + x_1m^2]^{-3} \\ &= 2\int_0^1 dx\,(1{-}x)[q^2 + D]^{-3} \; , \end{aligned} \tag{62.52}$$

where $q = \ell + xp$ and $D = x(1{-}x)p^2 + xm^2$. We set $\ell = q - xp$ in the numerator, and drop terms that are odd in q. Then only the q^2 terms contribute to the divergence. These terms are $x(\not q\not q\not p + \not q\not p\not q + \not p\not q\not q) + \not q(-\not p + m)\not q$, and making the replacement $q^\mu q^\nu \to \frac{1}{4}q^2 g^{\mu\nu}$ yields $\frac{1}{4}q^2[x(\gamma^\mu\gamma_\mu\not p + \gamma^\mu\not p\gamma_\mu + \not p\gamma^\mu\gamma_\mu) + \gamma^\mu(-\not p + m)\gamma_\mu] = \frac{1}{4}q^2[x(-4\not p + 2\not p - 4\not p) - 2\not p - 4m] = -\frac{1}{4}q^2[(6x{+}2)\not p + 4m]$. (We can set $d = 4$ because terms of order ε will not contribute to the divergent part.) Then we use

$$\int \frac{d^4q}{(2\pi)^4} \frac{q^2}{(q^2 + D)^3} = \frac{i}{8\pi^2\varepsilon} + \text{finite} \tag{62.53}$$

and $2\int_0^1 dx\,(1{-}x) = 1$ and $2\int_0^1 dx\,(1{-}x)x = \frac{1}{3}$ to get

$$\Delta\Sigma(p) = -(\xi{-}1)\frac{e^2}{8\pi^2\varepsilon}(\not p + m) - (\Delta Z_2)\not p - (\Delta Z_m)m \; , \tag{62.54}$$

Combining this with eqs. (62.34) and (62.35), we get

$$Z_2 = 1 - \xi\frac{e^2}{8\pi^2}\left(\frac{1}{\varepsilon} + \text{finite}\right) + O(e^4) \; , \tag{62.55}$$

$$Z_m = 1 - (3{+}\xi)\frac{e^2}{8\pi^2}\left(\frac{1}{\varepsilon} + \text{finite}\right) + O(e^4) \; . \tag{62.56}$$

To compute the change in Z_1, we can set external momenta to zero. Then we have

$$i\Delta V^\mu(0,0) = ie\Delta Z_1\gamma^\mu + (\xi-1)e^3 \int \frac{d^4\ell}{(2\pi)^4} \frac{\not\ell(-\not\ell+m)\gamma^\mu(-\not\ell+m)\not\ell}{(\ell^2+m^2)^2(\ell^2)^2} \ . \tag{62.57}$$

Only the ℓ^4 term in the numerator gives a divergence, so we can replace the numerator with $\not\ell\not\ell\gamma^\mu\not\ell\not\ell = (\ell^2)^2\gamma^\mu$. The divergent part of the integral is then $i/8\pi^2\varepsilon$, and so the divergent part of ΔZ_1 is $-(\xi-1)e^2/8\pi^2\varepsilon$, leading to

$$Z_1 = 1 - \xi\frac{e^2}{8\pi^2}\left(\frac{1}{\varepsilon} + \text{finite}\right) + O(e^4) \ . \tag{62.58}$$

We see that $Z_1 = Z_2$ for all ξ, and that $Z_1 = Z_2 = 1 + O(e^4)$ for Lorenz gauge ($\xi = 0$). This will prove very convenient later.

62.3) The diagrams consist of a closed fermion loop with four external photons. Starting with photon #1 and following the fermion arrow backwards, there are six diagrams, corresponding to the six permutations of $2, 3, 4$. To get a divergent result, we must keep all the loop momenta in the numerator. The divergent part of the diagram with 1234 ordering is then

$$i\mathcal{T}_{\text{div}} = e^4 \int \frac{d^4\ell}{(2\pi)^4} \frac{\text{Tr}\,\not\varepsilon_1\not\ell\not\varepsilon_2\not\ell\not\varepsilon_3\not\ell\not\varepsilon_4\not\ell}{(\ell^2+m^2)^4} \tag{62.59}$$

Using symmetric integration, we have $\ell^\mu\ell^\nu\ell^\rho\ell^\sigma \to \frac{1}{24}(\ell^2)^2(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$, and so $\text{Tr}\,\not\varepsilon_1\not\ell\not\varepsilon_2\not\ell\not\varepsilon_3\not\ell\not\varepsilon_4\not\ell \to \frac{1}{24}(\ell^2)^2\,\text{Tr}(\not\varepsilon_1\gamma^\mu\not\varepsilon_2\gamma_\mu\not\varepsilon_3\gamma^\rho\not\varepsilon_4\gamma_\rho + \not\varepsilon_1\gamma^\mu\not\varepsilon_2\gamma^\nu\not\varepsilon_3\gamma_\mu\not\varepsilon_4\gamma_\nu + \not\varepsilon_1\gamma^\mu\not\varepsilon_2\gamma^\nu\not\varepsilon_3\gamma_\nu\not\varepsilon_4\gamma_\mu)$. The first and last term in the parentheses each simplifies to $4\not\varepsilon_1\not\varepsilon_2\not\varepsilon_3\not\varepsilon_4$, while the middle term becomes $2\not\varepsilon_1\not\varepsilon_3\gamma^\nu\not\varepsilon_2\not\varepsilon_4\gamma_\nu = 8(\varepsilon_2\varepsilon_4)\not\varepsilon_1\not\varepsilon_3$. Taking the trace then yields $\text{Tr}(\ldots) = 32[(\varepsilon_1\varepsilon_2)(\varepsilon_3\varepsilon_4) + (\varepsilon_1\varepsilon_4)(\varepsilon_2\varepsilon_3) - 2(\varepsilon_1\varepsilon_3)(\varepsilon_2\varepsilon_4)]$. If we now sum over the six permutations of 234, the terms cancel in pairs, and the result is zero.

If the result were not zero, we would have to add a term to the lagrangian to absorb the divergence. Since no external momenta are involved, such a term would have to take the form $A^\mu A_\mu A^\nu A_\nu$. However, this is not gauge invariant. Thus gauge invariance requires that $\mathcal{T}_{\text{div}}$ vanish.

63 The Vertex Function in Spinor Electrodynamics

63.1) a) If we have an incoming electron with momentum p and an outgoing electron with momentum p' that are attached to the same vertex, then we get a factor of $\overline{u}'\mathbf{V}^\mu u$, where $u = u_s(\mathbf{p})$ and $\overline{u}' = \overline{u}_{s'}(\mathbf{p}')$. The photon momentum is $q = p' - p$. Since $q^\mu q^\nu$ terms in the photon propagator $\mathbf{\Delta}^{\mu\nu}(q)$ should not contribute, and since the photon propagator attaches to the vertex $\mathbf{V}^\mu(p', p)$, we should have $q_\mu \overline{u}'\mathbf{V}^\mu u = 0$. Using eq. (63.23), we get

$$\begin{aligned} 0 &= e(p'-p)_\mu \overline{u}'[A\gamma^\mu + B(p'+p)^\mu + C(p'-p)^\mu]u \\ &= e\overline{u}'[A(\not{p}' - \not{p}) + B(p'^2 - p^2) + Cq^2]u \\ &= eCq^2\overline{u}'u\ , \end{aligned} \tag{63.24}$$

where we used $\overline{u}'\not{p}' = -m\overline{u}'$, $\not{p}u = -mu$, and $p'^2 = p^2 = -m^2$ to get the last line. We see that we must have $C(q^2) = 0$.

b) Using eq. (63.16), we can make the replacement $A\gamma^\mu + B(p'+p)^\mu \rightarrow (A + 2mB)\gamma^\mu + 2iBS^{\mu\nu}q_\nu$. Comparing with eq. (63.18), we see that $F_1 = A + 2mB$ and $F_2 = -2mB$.

64 The Magnetic Moment of the Electron

64.1) It is easiest to use a different gauge for the external field, $\mathbf{A} = \frac{1}{2}B(-y, x, 0)$ rather than $\mathbf{A} = B(0, x, 0)$. Then, in eq. (64.10), the $i\gamma^2\partial_1$ term (where $\partial_1 \equiv \partial/\partial p_1$) becomes $\frac{1}{2}i(\gamma^2\partial_1 - \gamma^1\partial_2)$. Using $\overline{u}\gamma^i u = 2p^i\overline{u}u$ and $\overline{u}u = 2m$, this becomes $-\frac{i}{2m}(p^1\partial_2 - p^2\partial_1)$, which we recognize as $\frac{1}{2m}L_z$ acting on functions of $\mathbf{p}$. Comparing with the result in eq. (64.13), we see that $(1+\alpha/2\pi)S_z$ is replaced with $\frac{1}{2}L_z + (1+\alpha/2\pi)S_z$. Then L_z is replaced by its eigenvalue m_ℓ, and S_z by its eigenvalue $m_s = +\frac{1}{2}$.

65 Loop Corrections in Scalar Electrodynamics

65.1) For vanishing photon four-momenta, and external scalars on shell (that is, $k^2 = k'^2 = k\cdot p = k\cdot p' = k'\cdot p = k'\cdot p' = 0$, $p^2 = p'^2 = -m^2$), we have $\mathbf{V}_3^\mu = -e(p+p')$ and $\mathbf{V}_4^{\mu\nu} = -2e^2 g^{\mu\nu}$.

65.2) Define a covariant derivative $D^\mu \equiv \partial^\mu - iKeA^\mu$, where K is an arbitrary constant. Our results in section 58 imply that, under the transformation $A^\mu \to A^\mu - \partial^\mu\Gamma$ and $\varphi \to e^{-iKe\Gamma}\varphi$, we have $D^\mu\varphi \to e^{-iKe\Gamma}D^\mu\varphi$. Then $-(D^\mu\varphi)^\dagger D_\mu\varphi = -\partial^\mu\varphi^\dagger\partial_\mu\varphi + iKe[\varphi^\dagger\partial^\mu\varphi - \varphi^\dagger(\partial^\mu\varphi)]A^\mu - K^2e^2\varphi^\dagger\varphi A^\mu A_\mu$ is invariant. If we multiply by Z_2 and compare with eqs. (65.1–4), we see that we must have $Z_1 = KZ_2$ and $Z_4 = K^2Z_2$. Eliminating K yields $Z_4 = Z_1^2/Z_2$.

66 Beta Functions in Quantum Electrodynamics

66.1) We have

$$\begin{aligned}\gamma_\Psi &\equiv \frac{1}{2}\frac{d\ln Z_2}{d\ln\mu} \\ &= \frac{1}{2}\left(-\frac{1}{2}\,e\frac{\partial}{\partial e}\right)\varepsilon\ln Z_2 \\ &= \frac{e^2}{16\pi^2}\;, \end{aligned} \tag{66.33}$$

$$\begin{aligned}\gamma_A &\equiv \frac{1}{2}\frac{d\ln Z_3}{d\ln\mu} \\ &= \frac{1}{2}\left(-\frac{1}{2}\,e\frac{\partial}{\partial e}\right)\varepsilon\ln Z_3 \\ &= \frac{e^2}{12\pi^2}\;, \end{aligned} \tag{66.34}$$

$$\begin{aligned}\gamma_m &\equiv \frac{d}{d\ln\mu}\ln m \\ &= \frac{d}{d\ln\mu}\Big[\ln m_0 - \ln\Big(Z_m/Z_2\Big)\Big] \\ &= \left(\frac{1}{2}\,e\frac{\partial}{\partial e}\right)\varepsilon\ln\Big(Z_m/Z_2\Big) \\ &= -\frac{3e^2}{8\pi^2}\;. \end{aligned} \tag{66.35}$$

66.2) We have

$$\begin{aligned}\gamma_\varphi &\equiv \frac{1}{2}\frac{d\ln Z_2}{d\ln\mu} \\ &= \frac{1}{2}\left(-\frac{1}{2}\,e\frac{\partial}{\partial e} - \lambda\,\frac{\partial}{\partial\lambda}\right)\varepsilon\ln Z_2 \\ &= -\frac{3e^2}{16\pi^2}\;, \end{aligned} \tag{66.36}$$

$$\begin{aligned}\gamma_A &\equiv \frac{1}{2}\frac{d\ln Z_3}{d\ln\mu} \\ &= \frac{1}{2}\left(-\frac{1}{2}\,e\frac{\partial}{\partial e} - \lambda\,\frac{\partial}{\partial\lambda}\right)\varepsilon\ln Z_3 \\ &= \frac{e^2}{48\pi^2}\;, \end{aligned} \tag{66.37}$$

$$
\begin{aligned}
\gamma_m &\equiv \frac{d}{d\ln\mu}\ln m \\
&= \frac{d}{d\ln\mu}\Big[\ln m_0 - \ln\Big(Z_m^{1/2}/Z_2^{1/2}\Big)\Big] \\
&= \left(\frac{1}{2}\,e\frac{\partial}{\partial e} + \lambda\,\frac{\partial}{\partial\lambda}\right)\varepsilon\ln\Big(Z_m^{1/2}/Z_2^{1/2}\Big) \\
&= \frac{1}{16\pi^2}(\lambda - 3e^2)\;. \qquad (66.38)
\end{aligned}
$$

66.3) We have $Z_3 = 1 - e^2/(6\pi^2\epsilon)$, $Z_1 = Z_2 = 1 - \xi e^2/(8\pi^2\epsilon)$, and $Z_m = 1 - (3{+}\xi)e^2/(8\pi^2\epsilon)$. We see that Z_3 is independent of ξ (to this order), and also that dependence on ξ cancels in the ratios Z_1/Z_2 and Z_m/Z_2. Since the beta function is computed from $Z_1/(Z_2 Z_3^{1/2})$ and the anomalous dimension of m from Z_m/Z_2, these are independent of ξ. The results are therefore the same as in problem 66.1.

66.4) $1/\alpha(M_{\rm W}) = 137.036 - (2/3\pi)(40.07) = 128.5$. The measured value of $1/\alpha(M_{\rm Z})$ is 127.9.

67 Ward Identities in Quantum Electrodynamics I

67.1) Making the replacement $\varepsilon_{1'} \to k_1'$, the amplitude becomes

$$\mathcal{T} = -e^2 \left[\frac{4(k_1 \cdot k_{1'})(k_2 \cdot \varepsilon_{2'})}{m^2 - t} + \frac{4(k_1 \cdot \varepsilon_{2'})(k_2 \cdot k_{1'})}{m^2 - u} + 2(k_{1'} \cdot \varepsilon_{2'}) \right] . \tag{67.13}$$

Now we use $k_1 \cdot k_{1'} = \frac{1}{2}(t - m^2)$ and $k_2 \cdot k_{1'} = \frac{1}{2}(u - m^2)$ to get $\mathcal{T} = 2e^2(k_2 + k_1 - k_1') \cdot \varepsilon_{2'} = 2e^2\, k_2' \cdot \varepsilon_{2'} = 0$.

67.2) Making the replacement $\varepsilon_{1'} \to k_1'$, and using $-p_1 + k_2' = p_2 - k_1'$ in the numerator of the second term, the amplitude becomes

$$\mathcal{T} = e^2\, \overline{v}_2 \left[\not{\varepsilon}_{2'} \left(\frac{-\not{p}_1 + \not{k}_1' + m}{m^2 - t} \right) \not{k}_1' + \not{k}_1' \left(\frac{\not{p}_2 - \not{k}_1' + m}{m^2 - u} \right) \not{\varepsilon}_{2'} \right] u_1 \, . \tag{67.14}$$

We use $\not{k}_1' \not{k}_1' = -k_1'^{\,2} = 0$ to remove the $\not{k}_1'$ term in each numerator. Then we use $(-\not{p}_1 + m)\not{k}_1' = \not{k}_1'(\not{p}_1 + m) + 2p_1 \cdot k_1'$ in the first term and $\not{k}_1'(\not{p}_2 + m) = (-\not{p}_2 + m)\not{k}_1' - 2p_2 \cdot k_1'$ in the second term, followed by $(\not{p}_1 + m)u_1 = 0$ in the first term and $\overline{v}_2(-\not{p}_2 + m) = 0$ in the second term. Then using $2p_1 \cdot k_1' = t - m^2$ and $2p_2 \cdot k_1' = u - m^2$, we see that the two terms cancel.

68 Ward Identities in Quantum Electrodynamics II

68.1) a) Consider the photon propagator, which (in momentum space) can be expressed as $\tilde{\mathbf{\Delta}}^{\mu\nu}(k) = \Delta^{\mu\nu}(k) + \tilde{\Delta}^{\mu\rho}(k)\Pi_{\rho\sigma}(k)\tilde{\Delta}^{\sigma\nu}(k) + \ldots$. All terms except the first consist of Feynman diagrams with two external photons, each attached to a vertex. If drop the first term and remove the external photons from the remaining terms, we get the sum of all diagrams with two vertices that have no attached photons; each of these two vertices corresponds (in position space) to a factor of $Z_1 j^\mu(x)$. We conclude that $\Pi^{\mu\nu}(k) + \Pi^{\mu\rho}(k)\tilde{\Delta}_{\rho\sigma}(k)\Pi^{\sigma\nu}(k) + \ldots$ is proportional to the Fourier transform of $\langle 0|\mathrm{T}j^\mu(x)j^\nu(y)|0\rangle$.

b) We have $\partial_\mu\langle 0|\mathrm{T}j^\mu(x)j^\nu(y)|0\rangle = 0$ by the Ward identity. In momentum space, this becomes $k_\mu\Pi^{\mu\rho}(k)[\delta_\rho{}^\nu + \tilde{\Delta}_{\rho\sigma}(k)\Pi^{\sigma\nu}(k) + \ldots] = 0$. The matrix in square brackets is nonzero in general, and so the entire expression can vanish only if $k_\mu\Pi^{\mu\rho}(k) = 0$.

68.2) a) From eq. (62.28), we have

$$\Sigma_{1\,\mathrm{loop}}(\not{p}) = -ie^2 \int \frac{d^4\ell}{(2\pi)^4}\, \gamma^\rho \tilde{S}(\not{p}+\not{\ell})\gamma^\nu \tilde{\Delta}_{\nu\rho}(\ell)\ . \tag{68.18}$$

From eq (62.40), we have

$$\mathbf{V}^\mu_{1\,\mathrm{loop}}(p',p) = -ie^3 \int \frac{d^4\ell}{(2\pi)^4}\gamma^\rho \tilde{S}(\not{p}'+\not{\ell})\gamma^\mu \tilde{S}(\not{p}+\not{\ell})\gamma^\nu \tilde{\Delta}_{\nu\rho}(\ell)\ . \tag{68.19}$$

Contract this with $(p'-p)^\mu$, and use $\not{p}'-\not{p} = (\not{p}'+\not{\ell}+m) - (\not{p}+\not{\ell}+m) = \tilde{S}(\not{p}'+\not{\ell})^{-1} - \tilde{S}(\not{p}+\not{\ell})^{-1}$ to get

$$\begin{aligned}(p'-p)_\mu \mathbf{V}^\mu_{1\,\mathrm{loop}}(p',p) &= -ie^3 \int \frac{d^4\ell}{(2\pi)^4}\gamma^\rho[\tilde{S}(\not{p}+\not{\ell}) - \tilde{S}(\not{p}'+\not{\ell})]\gamma^\nu \tilde{\Delta}_{\nu\rho}(\ell) \\ &= e\Sigma_{1\,\mathrm{loop}}(\not{p}) - e\Sigma_{1\,\mathrm{loop}}(\not{p}')\ .\end{aligned} \tag{68.20}$$

We have $V^\mu(p',p) = Z_1 e\gamma^\mu + \mathbf{V}^\mu_{1\,\mathrm{loop}}(p',p)$ and $\tilde{S}(\not{p})^{-1} = Z_2\not{p} + Z_m m - \Sigma_{1\,\mathrm{loop}}(\not{p}')$. Assuming $Z_1 = Z_2$, we have $(p'-p)_\mu(Z_1 e\gamma^\mu) = e[(Z_2\not{p}' + Z_m m) - (Z_2\not{p} + Z_m m)]$. Combining this with eq. (68.20) yields $(p'-p)_\mu \mathbf{V}^\mu(p',p) = e[\tilde{S}(\not{p}')^{-1} - \tilde{S}(\not{p})^{-1}]$ up through one-loop in any scheme where $Z_1 = Z_2$.

68.3) An insertion of $-iZ_1 e[\varphi^\dagger\partial^\mu\varphi - (\partial^\mu\varphi^\dagger)\varphi]$ produces a photon-scalar-scalar vertex, without the photon. An insertion of $-2iZ_4 e^2 A^\mu\varphi^\dagger\varphi$ produces a photon-photon-scalar-scalar vertex, without one of the photons. The sum of these equals $Z_2^{-1}Z_1 J^\mu$, where J^μ is the Noether current. (We need to use $Z_4 = Z_1^2/Z_2$ to get this result.) Thus, the correlation function

$$C_3^\mu(k,p',p) \equiv iZ_2^{-1}Z_1 \int d^4x\, d^4y\, d^4z\, e^{ikx-ip'y+ipz}\,\langle 0|\mathrm{T}J^\mu(x)\varphi(y)\varphi^\dagger(z)|0\rangle\ , \tag{68.21}$$

can be expressed in terms of the exact scalar propagator $\tilde{\mathbf{\Delta}}(p)$ and the exact photon–scalar–scalar vertex function $\mathbf{V}_3^\mu(p',p)$ as

$$C_3^\mu(k,p',p) = (2\pi)^4\delta^4(k+p-p')\Big[\tfrac{1}{i}\tilde{\mathbf{\Delta}}(p')i\mathbf{V}_3^\mu(p',p)\tfrac{1}{i}\tilde{\mathbf{\Delta}}(p)\Big]\ . \tag{68.22}$$

There are contributions to $\mathbf{V}_3$ from diagrams where an internal photon attaches to the same vertex as the external photon; these are generated by the second term in the Noether current. We now use the Ward identity, eq. (68.4), along with $\delta\varphi = -ie\varphi$ and $\delta\varphi^\dagger = +ie\varphi^\dagger$, to get

$$-\partial_\mu\langle 0|\mathrm{T}J^\mu(x)\varphi(y)\varphi^\dagger(z)|0\rangle = +e\delta^4(x-y)\langle 0|\mathrm{T}\varphi(y)\varphi^\dagger(z)|0\rangle - e\delta^4(x-z)\langle 0|\mathrm{T}\varphi(y)\varphi^\dagger(z)|0\rangle \ . \tag{68.23}$$

From here the analysis is essentially identical to that of spinor electrodynamics, and we get

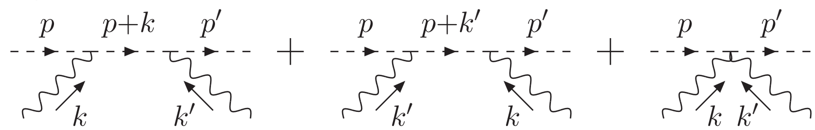
$$(p'-p)_\mu \mathbf{V}_3^\mu(p',p) = Z_2^{-1}Z_1 e\Big[\tilde{\mathbf{\Delta}}(p')^{-1} - \tilde{\mathbf{\Delta}}(p)^{-1}\Big] \ . \tag{68.24}$$

b) Since both $\mathbf{V}_3$ and $\tilde{\mathbf{\Delta}}$ are finite, Z_1/Z_2 must be finite as well. Since all corrections to $Z_i = 1$ are infinite in the $\overline{\text{MS}}$ scheme, eq. (68.24) is consistent only if $Z_1 = Z_2$. In the OS scheme, we use the fact that near $p^2 = p'^2 = -m^2$ and $(p'-p)^2 = 0$, $\tilde{\mathbf{\Delta}}(p)^{-1} \sim p^2 + m^2$ and $\mathbf{V}_3^\mu(p',p) \sim -e(p'+p)^\mu$ to see that we must have $Z_1 = Z_2$.

c) We define

$$C_4^{\mu\nu}(k,k',p',p) \equiv i^2 Z_2^{-2} Z_1^2 \int d^4x\, d^4y\, d^4z\, d^4w\, e^{ikx+ik'w-ip'y+ipz}\langle 0|\mathrm{T}J^\mu(x)J^\nu(w)\varphi(y)\varphi^\dagger(z)|0\rangle \ . \tag{68.25}$$

This gets contributions from the exact three- and four-point vertices:



We have

$$\begin{aligned} C_4^{\mu\nu}(k,k',p',p) \;=\; & (2\pi)^4\delta^4(p+k+k'-p')\tfrac{1}{i}\tilde{\mathbf{\Delta}}(p') \\ & \times \Big[i\mathbf{V}_3^\nu(p',p+k)\tfrac{1}{i}\tilde{\mathbf{\Delta}}(p+k)i\mathbf{V}_3^\mu(p+k,p) \\ & + i\mathbf{V}_3^\mu(p',p+k')\tfrac{1}{i}\tilde{\mathbf{\Delta}}(p+k')i\mathbf{V}_3^\nu(p+k',p) \\ & + i\mathbf{V}_4^{\mu\nu}(k,k',p',p)\Big]\tfrac{1}{i}\tilde{\mathbf{\Delta}}(p) \ . \end{aligned} \tag{68.26}$$

Multiplying by k_μ, and following the steps that lead to eq. (68.3), we get

$$\begin{aligned} k_\mu C^{\mu\nu}(k,k',p',p) \;=\; & -iZ_2^{-2}Z_1^2\int d^4x\, d^4y\, d^4z\, d^4w\, e^{ikx-ip'y+ipz-ik'w} \\ & \times \partial_\mu\langle 0|\mathrm{T}J^\mu(x)J^\nu(w)\varphi(y)\varphi^\dagger(z)|0\rangle \ . \end{aligned} \tag{68.27}$$

The relevant Ward idenitity is

$$\begin{aligned} -\partial_\mu\langle 0|\mathrm{T}J^\mu(x)J^\nu(w)\varphi(y)\varphi^\dagger(z)|0\rangle \;=\; & +e\delta^4(x-y)\langle 0|\mathrm{T}J^\nu(w)\varphi(y)\varphi^\dagger(z)|0\rangle \\ & -e\delta^4(x-z)\langle 0|\mathrm{T}J^\nu(w)\varphi(y)\varphi^\dagger(z)|0\rangle \ . \end{aligned} \tag{68.28}$$

There is no $\delta^4(x-w)$ term because $J^\nu(w)$ is invariant under the U(1) symmetry. Plugging eq. (68.28) into eq. (68.27) and using eq. (68.21), we get

$$k_\mu C_4^{\mu\nu}(k,k',p',p) \;=\; Z_2^{-1}Z_1 e\Big[C_3^\nu(k',p'-k,p) - C_3^\nu(k',p',p+k)\Big] \ . \tag{68.29}$$

Now we evaluate the left-hand side of eq. (68.29), using eq. (68.26) and then simplifying with eq. (68.24). We also use eq. (68.22) on the right-hand side of eq. (68.29). Then, after some rearranging and use of $p+k = p'-k'$, we find

$$k_\mu \mathbf{V}_4^{\mu\nu}(k,k',p',p) = Z_2^{-1} Z_1 e \Big[\mathbf{V}_3^\mu(p+k',p) - \mathbf{V}_3^\mu(p',p'-k') \Big] \,. \tag{68.30}$$

Since $Z_2^{-1} Z_1 = Z_1^{-1} Z_4$, this is the same as eq. (68.17).

69 Nonabelian Gauge Theory

69.1) For $A_\mu = A^b_\mu T^b_{\rm R}$ and $U = I - ig\theta^a T^a_{\rm R}$, eq. (69.9) becomes

$$\begin{aligned} A^b_\mu T^b_{\rm R} &\to A^b_\mu T^b_{\rm R} - ig\theta^a A^b_\mu [T^a_{\rm R}, T^b_{\rm R}] - \partial_\mu \theta^a T^a_{\rm R} \\ &= A^b_\mu T^b_{\rm R} + g\theta^a A^b_\mu f^{abe} T^e_{\rm R} - \partial_\mu \theta^a T^a_{\rm R} \,, \end{aligned} \tag{69.26}$$

or equivalently

$$A^e_\mu T^e_{\rm R} \to (A^e_\mu + g\theta^a A^b_\mu f^{abe} - \partial_\mu \theta^e) T^e_{\rm R} \,. \tag{69.27}$$

We mutiply by $T^d_{\rm R}$, take the trace, and use $\mathrm{Tr}\, T^d_{\rm R} T^e_{\rm R} = C^{de}$, where C^{de} is a positive-definite, real symmetric (and hence invertible) matrix. Then we matrix-multiply by $(C^{-1})^{cd}$ to get

$$A^c_\mu \to A^c_\mu + g\theta^a A^b_\mu f^{abc} - \partial_\mu \theta^c \,, \tag{69.28}$$

which is independent of the representation. (We can always choose the generators so that $C^{de} \propto \delta^{de}$, which makes the final step superfluous.)

69.2) $[T^a T^a, T^b] = [T^a, T^b] T^a + T^a [T^a, T^b] = i f^{abc} (T^c T^a + T^a T^c)$. Since f^{abc} is antisymmetric on $a \leftrightarrow c$, while $T^c T^a + T^a T^c$ is symmetric, the result is zero.

70 Group Representations

70.1) Contracting $(T^a_{\rm R}T^b_{\rm R})_{ii} = T({\rm R})\delta^{ab}$ with δ^{ab} yields $(T^a_{\rm R}T^a_{\rm R})_{ii} = T({\rm R})\delta^{aa} = T({\rm R})D({\rm A})$. Contracting $(T^a_{\rm R}T^a_{\rm R})_{ij} = C({\rm R})\delta_{ij}$ with δ_{ij} yields $(T^a_{\rm R}T^a_{\rm R})_{ii} = C({\rm R})\delta_{ii} = C({\rm R})D({\rm R})$.

70.2) a) $T({\rm N}\otimes\overline{\rm N}) = T({\rm A})+T(1) = T({\rm A})+0 = T({\rm A})$, and $T({\rm N}\otimes\overline{\rm N}) = T({\rm N})D(\overline{\rm N})+D({\rm N})T(\overline{\rm N}) = \frac{1}{2}N + N\frac{1}{2} = N$.

b) For SU(2), $T(3)\delta^{ab} = (T^a_3)^{cd}(T^b_3)^{dc} = (-i)^2\varepsilon^{acd}\varepsilon^{bdc} = \varepsilon^{acd}\varepsilon^{bcd}$. This vanishes if $a \neq b$, since then there is no way to get both epsilons to be nonzero; if $a = b = 1$ (say), then $\varepsilon^{1cd}\varepsilon^{1cd} = \varepsilon^{123}\varepsilon^{123} + \varepsilon^{132}\varepsilon^{132} = 1 + 1 = 2$. So $T({\rm A}) = 2$.

c) ${\rm N}\otimes\overline{\rm N} = (2 \oplus (N{-}2)1\text{'s}) \otimes (2 \oplus (N{-}2)1\text{'s}) = (2 \otimes 2) \oplus (2N{-}4)2\text{'s} \oplus (N{-}2)^2 1\text{'s}$. Using $2 \otimes 2 = 3 \oplus 1$ and ${\rm N}\otimes\overline{\rm N} = {\rm A} \oplus 1$, we have ${\rm A} = 3 \oplus (2N{-}4)2\text{'s} \oplus (N{-}2)^2 1\text{'s}$.

d) $T({\rm A}) = T(3) + (2N{-}4)T(2) + (N{-}1)^2 T(1) = 2 + (2N{-}4)\frac{1}{2} + 0 = N$.

70.3) a) ${\rm A} = [{\rm N}\otimes{\rm N}]_{\rm A} = [(3\oplus(N{-}3)1\text{'s})\otimes(3\oplus(N{-}3)1\text{'s})]_{\rm A} = [3\otimes3]_{\rm A}\oplus(N{-}3)3\text{'s} = 3\oplus(N{-}3)3\text{'s} = (N{-}2)3\text{'s}$.

b) $T({\rm A}) = (N{-}2)T(3) = (N{-}2)2 = 2N{-}4$.

70.4) a) $D(\mathcal{A}) = \frac{1}{2}N(N{-}1)$ and $D(\mathcal{S}) = \frac{1}{2}N(N{+}1)$.

b) $\mathcal{A} = [{\rm N}\otimes{\rm N}]_{\rm A} = [(2\oplus(N{-}2)1\text{'s})\otimes(2\oplus(N{-}2)1\text{'s}]_{\rm A} = [2\otimes2]_{\rm A}\oplus(N{-}2)2\text{'s} = 1\oplus(N{-}2)2\text{'s}$. Therefore $T(\mathcal{A}) = T(1)+(N{-}2)T(2) = 0+(N{-}2)\frac{1}{2} = \frac{1}{2}(N{-}2)$. Similarly, $\mathcal{S}\oplus1 = [{\rm N}\otimes{\rm N}]_{\rm S} = [(2\oplus(N{-}2)1\text{'s})\otimes(2\oplus(N{-}2)1\text{'s})]_{\rm S} = [2\otimes2]_{\rm S}\oplus(N{-}2)2\text{'s}\oplus(N{-}2)^2 1\text{'s} = 3\oplus(N{-}2)2\text{'s}\oplus(N{-}2)^2 1\text{'s}$. Therefore $T(\mathcal{S}) = T(3) + (N{-}2)T(2) = 2 + (N{-}2)\frac{1}{2} = \frac{1}{2}(N{+}2)$.

c) $\varphi_{ij} = \varepsilon_{ijk}\varphi^k$.

d) $\mathcal{A} = [{\rm N}\otimes{\rm N}]_{\rm A} = [(3\oplus(N{-}3)1\text{'s})\otimes(3\oplus(N{-}3)1\text{'s}]_{\rm A} = [3\otimes3]_{\rm A}\oplus(N{-}3)3\text{'s} = \overline{3}\oplus(N{-}3)3\text{'s}$. Therefore $A(\mathcal{A}) = A(\overline{3}) + (N{-}3)A(3) = -1 + (N{-}3)(+1) = N{-}4$. Similarly, $\mathcal{S} = [{\rm N}\otimes{\rm N}]_{\rm S} = [(3\oplus(N{-}3)1\text{'s})\otimes(3\oplus(N{-}3)1\text{'s}]_{\rm S} = [3\otimes3]_{\rm S}\oplus(N{-}3)3\text{'s}\oplus(N{-}3)^2 1\text{'s} = 6\oplus(N{-}3)3\text{'s}\oplus(N{-}3)^2 1\text{'s}$. Therefore $A(\mathcal{S}) = A(6) + (N{-}3)A(3) + (N{-}3)^2 A(1) = A(6) + (N{-}3)(+1) + 0 = A(6)+N{-}3$. To compute $A(6)$ for SU(3), we note that $A(3\otimes3) = D(3)A(3) + A(3)D(3) = 6$, and that $A(3\otimes3) = A(6\oplus\overline{3}) = A(6) + A(\overline{3}) = A(6) - 1$, so $A(6) = 7$. Therefore for SU(N), $A(\mathcal{S}) = N{+}4$.

70.5) a) $[D_\mu(\varphi\chi)]_{iI} = \partial_\mu(\varphi_i\chi_I) - igA^a_\mu(T^a_{{\rm R}_1\otimes{\rm R}_2})_{iI,jJ}\varphi_j\chi_J$, and $(T^a_{{\rm R}_1\otimes{\rm R}_2})_{iI,jJ} = (T_{{\rm R}_1})_{ij}\delta_{IJ} + \delta_{ij}(T_{{\rm R}_2})_{IJ}$, so $(T^a_{{\rm R}_1\otimes{\rm R}_2})_{iI,jJ}\varphi_j\chi_J = (T^a_{{\rm R}_1}\varphi)_i\chi_I + \varphi_i(T^a_{{\rm R}_2}\chi)_J$. Combining this with $\partial_\mu(\varphi_i\chi_I) = (\partial_\mu\varphi_i)\chi_I + \varphi_i\partial_\mu\chi_I$, we get $[D_\mu(\varphi\chi)]_{iI} = (D_\mu\varphi)_i\chi_I + \varphi_i(D_\mu\chi)_I$.

b) We begin with $(D_\mu\varphi)_i = \partial_\mu\varphi_i - igA^a_\mu(T^a_{\rm R})_i{}^j\varphi_j$ and $(D_\mu\varphi^\dagger)^i = \partial_\mu\varphi^{\dagger i} - igA^a_\mu(T^a_{\overline{\rm R}})^i{}_k\varphi^{\dagger k} = \partial_\mu\varphi^{\dagger i} + igA^a_\mu(T^a_{\rm R})_k{}^i\varphi^{\dagger k}$, so $\varphi^{\dagger i}(D_\mu\varphi)_i = \varphi^{\dagger i}\partial_\mu\varphi_i - igA^a_\mu\varphi^{\dagger i}(T^a_{\rm R})_i{}^j\varphi_j$ and $(\mathcal{D}_\mu\varphi^\dagger)^i\varphi_i = (\partial_\mu\varphi^\dagger)^i + igA^a_\mu(T^a_{\rm R})_k{}^i\varphi^{\dagger k}\varphi_i$. Adding, the gauge-field terms cancel, and we get $\varphi^{\dagger i}(D_\mu\varphi)_i + (D_\mu\varphi^\dagger)^i\varphi_i = \varphi^{\dagger i}(\partial_\mu\varphi)_i + (\partial_\mu\varphi^\dagger)^i\varphi_i = \partial_\mu(\varphi^{\dagger i}\varphi_i)$. Since $\varphi^{\dagger i}\varphi_i$ is a singlet, $\partial_\mu(\varphi^{\dagger i}\varphi_i) = D_\mu(\varphi^{\dagger i}\varphi_i)$, so this is a special case of part (a).

70.6) Using $(T^c_{\rm A})^{ab} = -if^{cab}$, we have $(D_\rho F_{\mu\nu})^a = \partial_\rho F^a_{\mu\nu} - gf^{cab}A^c_\rho F^b_{\mu\nu} = \partial_\rho F^a_{\mu\nu} + gf^{abc}A^b_\rho F^c_{\mu\nu}$. Plugging in $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu$, we get

$$(D_\rho F_{\mu\nu})^a = \partial_\rho(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu) + gf^{abc}A^b_\rho(\partial_\mu A^c_\nu - \partial_\nu A^c_\mu + gf^{cde}A^d_\mu A^e_\nu)$$

$$
\begin{aligned}
&= \partial_\rho\partial_\mu A^a_\nu - \partial_\rho\partial_\nu A^a_\mu \\
&\quad + gf^{abc}(\partial_\rho A^b_\mu A^c_\nu + A^b_\mu\partial_\rho A^c_\nu + A^b_\rho\partial_\mu A^c_\nu - A^b_\rho\partial_\nu A^c_\mu) \\
&\quad + g^2 f^{abc}f^{cde}A^b_\rho A^d_\mu A^e_\nu \; . \qquad (70.42)
\end{aligned}
$$

We manipulate the first term on the second line via $\partial_\rho A^b_\mu A^c_\nu = A^c_\nu \partial_\rho A^b_\mu$ and $f^{abc}A^c_\nu\partial_\rho A^b_\mu = -f^{abc}A^b_\nu\partial_\rho A^c_\mu$. Then we have

$$
\begin{aligned}
(D_\rho F_{\mu\nu})^a &= \partial_\rho\partial_\mu A^a_\nu - \partial_\nu\partial_\rho A^a_\mu \\
&\quad + gf^{abc}(-A^b_\nu\partial_\rho A^c_\mu + A^b_\mu\partial_\rho A^c_\nu + A^b_\rho\partial_\mu A^c_\nu - A^b_\rho\partial_\nu A^c_\mu) \\
&\quad + g^2 f^{abc}f^{cde}A^b_\rho A^d_\mu A^e_\nu \; . \qquad (70.43)
\end{aligned}
$$

If we use eq. (70.43) in $D_\mu F_{\nu\rho} + D_\nu F_{\rho\mu} + D_\rho F_{\mu\nu}$, the terms from the first line of eq. (70.43) will cancel in pairs, as will the terms from the second line. Then the third line yields

$$
\begin{aligned}
(D_\mu F_{\nu\rho} + D_\nu F_{\rho\mu} + D_\rho F_{\mu\nu})^a &= g^2 f^{abc}f^{cde}(A^b_\mu A^d_\nu A^e_\rho + A^b_\nu A^d_\rho A^e_\mu + A^b_\rho A^d_\mu A^e_\nu) \\
&= g^2 f^{abc}f^{cde}(A^b_\mu A^d_\nu A^e_\rho + A^e_\mu A^b_\nu A^d_\rho + A^d_\mu A^e_\nu A^b_\rho) \\
&= g^2(f^{abc}f^{cde} + f^{adc}f^{ceb} + f^{aec}f^{cbd})A^b_\mu A^d_\nu A^e_\rho \; , \qquad (70.44)
\end{aligned}
$$

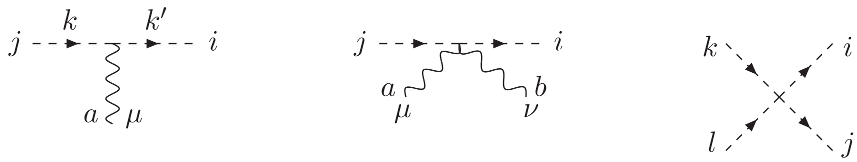
and the contents of the parentheses in the last line vanish by the Jacobi identity.

More elegant method: given a field φ^a in the adjoint representation, we can make a matrix-valued field $\varphi \equiv \varphi^a T^a$, analogous to the matrix-valued gauge field $A_\mu = A^a_\mu T^a$. Then the covariant derivative of φ, in matrix form, is $(D_\rho\varphi)^a T^a = \partial_\rho\varphi - ig[A_\rho,\varphi]$. We can write $\partial_\rho\varphi$ as a commutator $[\partial_\rho,\varphi]$; then we have $(D_\rho\varphi)^a T^a = [D_\rho,\varphi]$, where $D_\rho = \partial_\rho - igA_\rho$. Since $F_{\mu\nu}$ is a field in the adjoint representation, we have $(D_\rho F_{\mu\nu})^a T^a = [D_\rho, F_{\mu\nu}]$. We also have $F_{\mu\nu} = (i/g)[D_\mu, D_\nu]$, so $(D_\rho F_{\mu\nu})^a T^a = (i/g)[D_\rho,[D_\mu,D_\nu]]$. Adding the two cyclic permutations, the terms cancel in pairs when the commutators are written out. We see that the Bianchi identity is essentially the Jacobi identity applied to covariant derivatives.

71 The Path Integral for Nonabelian Gauge Theory

72 The Feynman Rules for Nonabelian Gauge Theory

72.1) This is a simple generalization of the vertices in scalar electrodynamics:



$ig(T^a)_{ij}(k+k')^\mu$ $\qquad -ig^2(T^aT^b+T^bT^a)_{ij}g^{\mu\nu}$ $\qquad -i\frac{1}{2}\lambda(\delta_{il}\delta_{jk}+\delta_{ik}\delta_{jl})$

73 The Beta Function in Nonabelian Gauge Theory

73.1) We need Z_1, Z_2, and Z_3. The computation of Z_2 and Z_3 are the same as in scalar electrodynamics, with extra group-theory factors. The extra group-theory factors are the same as they are for spinors. Thus we have $e^2 \to g^2 C(\mathrm{R})$ in Z_2 and $e^2 \to g^2 T(\mathrm{R})$ in Z_3. For Z_1, the two diagrams of fig. 73.2 contribute (with, obviously, the fermion line replaced by a scalar line). The first again gives the same result as scalar electrodynamics with $e^2 \to g^2 C(\mathrm{R})$. The second has a group theory factor that is proportional to $T(\mathrm{A})$. Thus it does not contribute to the dependence on the representation R of the scalar. The case of no scalars must give the same result as the case of no fermions (equivalent to R = 1), so we need not keep track of this diagram. The dependence on $C(\mathrm{R})$ then cancels in ratio Z_1/Z_2, and the g^2 term in Z_3 is smaller by a factor of four than it is for a fermion. Thus the contribution to the beta function is also smaller by the same factor, and so we have

$$\beta(g) = -\Big[\tfrac{11}{3}T(\mathrm{A}) - \tfrac{1}{3}T(\mathrm{R})\Big]\frac{g^3}{16\pi^2} + O(g^5)\ . \tag{73.42}$$

73.2) R-dependent contributions to Z_3 are additive, and R-dependent contributions to Z_1 and Z_2 cancel in the ratio. Thus we have

$$\beta(g) = -\Big[\tfrac{11}{3}T(\mathrm{A}) - \tfrac{4}{3}\textstyle\sum_i T(\mathrm{R_i}) - \tfrac{1}{3}\sum_i T(\mathrm{R}'_\mathrm{i})\Big]\dfrac{g^3}{16\pi^2} + O(g^5)\ . \tag{73.43}$$

73.3) Comparing with spinor electrodynamics, we have $e^2 \to n_\mathrm{F} T(\mathrm{R}) g^2$ in Z_3, and $e^2 \to C(\mathrm{R}) g^2$ in Z_2 and Z_m. Therefore we can use our results from problem 66.1 with the replacements $e^2 \to n_\mathrm{F} T(\mathrm{R}) g^2$ in γ_A and $e^2 \to C(\mathrm{R}) g^2$ in γ_Ψ and γ_m, so that

$$\gamma_\Psi = \frac{C(\mathrm{R})}{16\pi^2}\, g^2\ , \qquad \gamma_A = \frac{n_\mathrm{F} T(\mathrm{R})}{12\pi^2}\, g^2\ , \qquad \gamma_m = -\frac{3C(\mathrm{R})}{8\pi^2}\, g^2\ . \tag{73.44}$$

74 BRST Symmetry

74.1) We have $-i\int d^3x\, e^{ikx}\,\overleftrightarrow{\partial_0}\,A^\mu(x) = \sum_\lambda \varepsilon^\mu_\lambda(\mathbf{k})a^\dagger_\lambda(\mathbf{k})$. Contracting with $\varepsilon^*_{+\mu}(\mathbf{k})$ then yields $a^\dagger_+(\mathbf{k})$, since inspection of eq. (74.37) shows that $\varepsilon^*_{+\mu}(\mathbf{k})\varepsilon^\mu_\lambda(\mathbf{k}) = \delta_{+\lambda}$. On the other hand, contracting with ck_μ yields $-c\sqrt{2}\omega a^\dagger_<(\mathbf{k})$. According to eq. (74.40), $\sqrt{2}\omega a^\dagger_<(\mathbf{k}) = \xi\{Q_{\rm B}, b^\dagger(\mathbf{k})\}$, and so $|\chi\rangle = -c\xi b^\dagger(\mathbf{k})|\psi\rangle$.

75 Chiral Gauge Theories and Anomalies

75.1) We must demand that $\frac{1}{2}\mathrm{Tr}\{T^a,T^b\}T^c=0$, where T^a is a *either* a generator of the nonabelian group in the representation $\mathrm{R}_1\oplus\ldots\oplus\mathrm{R}_n$, *or* the generator Q of the abelian group. The nonabelian generators are block diagonal, with blocks given by $T^a_{\mathrm{R}_i}$, and Q is diagonal with $d(\mathrm{R}_1)$ entries Q_1, $d(\mathrm{R}_2)$ entries Q_2, etc. If all three generators are nonabelian, we have $\frac{1}{2}\mathrm{Tr}\{T^a,T^b\}T^c=\sum_i A(\mathrm{R}_i)d^{abc}$, and so we must have $\sum_i A(\mathrm{R}_i)=0$. If one generator (say T^c) is the abelian generator Q, we have $\frac{1}{2}\mathrm{Tr}\{T^a,T^b\}Q=\sum_i T(\mathrm{R}_i)Q_i\delta^{ab}$, and so we must have $\sum_i T(\mathrm{R}_i)Q_i=0$. If two generators (say T^a and T^b) are abelian, we have $\mathrm{Tr}\,Q^2T^c=\sum_i Q_i^2\,\mathrm{Tr}\,T^c_{\mathrm{R}_i}=0$, since nonabelian generators are always traceless. If all three generators are abelian, we have $\sum_i d(\mathrm{R}_i)Q_i^3$, and this must also vanish.

76 Anomalies in Global Symmetries

76.1) We have $A^\mu(x) = \sum_\lambda \int \widetilde{dk}\,[\varepsilon^{*\mu}_\lambda(\mathbf{k})a_\lambda(\mathbf{k})e^{ikx} + \varepsilon^\mu_\lambda(\mathbf{k})a^\dagger_\lambda(\mathbf{k})e^{-ikx}]$ in free field theory. Using $\langle p,q| = \langle 0|a_\lambda(\mathbf{p})a_{\lambda'}(\mathbf{q})$ then yields $\langle p,q|A_\nu(x)A_\sigma(y)|0\rangle = \varepsilon_\nu\varepsilon'_\sigma e^{-ipx-iqy} + \varepsilon'_\nu\varepsilon_\sigma e^{-iqx-ipy}$. Since $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$, we have

$$\begin{aligned}\langle p,q|F_{\mu\nu}(x)F_{\rho\sigma}(y)|0\rangle &= (-i)^2(p_\mu\varepsilon_\nu - p_\nu\varepsilon_\mu)(q_\rho\varepsilon'_\sigma - q_\sigma\varepsilon'_\rho)e^{-ipx-iqy} \\ &\quad + (-i)^2(q_\mu\varepsilon'_\nu - q_\nu\varepsilon'_\mu)(p_\rho\varepsilon_\sigma - p_\sigma\varepsilon_\rho)e^{-iqx-ipy}\;. \end{aligned} \tag{76.30}$$

Contracting with $\varepsilon^{\mu\nu\rho\sigma}$ yields

$$\varepsilon^{\mu\nu\rho\sigma}\langle p,q|F_{\mu\nu}(x)F_{\rho\sigma}(y)|0\rangle = -4\varepsilon^{\mu\nu\rho\sigma}(p_\mu\varepsilon_\nu q_\rho\varepsilon'_\sigma e^{-ipx-iqy} + q_\mu\varepsilon'_\nu p_\rho\varepsilon_\sigma e^{-iqx-ipy})\;. \tag{76.31}$$

Setting $x = y = z$ yields

$$\begin{aligned}\varepsilon^{\mu\nu\rho\sigma}\langle p,q|F_{\mu\nu}(z)F_{\rho\sigma}(z)|0\rangle &= -8\varepsilon^{\mu\nu\rho\sigma}p_\mu\varepsilon_\nu q_\rho\varepsilon'_\sigma e^{-i(p+q)z} \\ &= -8\varepsilon^{\mu\nu\rho\sigma}\varepsilon_\nu\varepsilon'_\sigma p_\mu q_\rho e^{-i(p+q)z} \\ &= +8\varepsilon^{\mu\nu\rho\sigma}\varepsilon_\mu\varepsilon'_\nu p_\rho q_\sigma e^{-i(p+q)z}\end{aligned} \tag{76.32}$$

Multiplying both sides of eq. (76.32) with $-g^2/16\pi^2$ and using eq. (76.14) yields eq. (76.29).

77 Anomalies and the Path Integral for Fermions

77.1) We begin by noting, for later use, that

$$\begin{aligned}\mathrm{Tr}(T^aT^bT^c) &= \tfrac{1}{2}\,\mathrm{Tr}(T^a[T^b,T^c]) + \tfrac{1}{2}\,\mathrm{Tr}(T^a\{T^b,T^c\}) \\ &= \tfrac{1}{2}iT(\mathrm{R})f^{abc} + A(\mathrm{R})d^{abc}\ .\end{aligned} \tag{77.37}$$

We have omitted the subscript R on the generators for notational convenience.

We will now show that each term on the right-hand side of eq. (77.35) is proportional to $A(\mathrm{R})$.

We note that $\varepsilon^{\mu\nu\rho\sigma}\partial_\mu(A_\nu\partial_\rho A_\sigma) = \varepsilon^{\mu\nu\rho\sigma}\partial_\mu A_\nu\partial_\rho A_\sigma$, since the term where ∂_μ and ∂_ρ both act on A_σ vanishes when contracted with $\varepsilon^{\mu\nu\rho\sigma}$. Now we have $\varepsilon^{\mu\nu\rho\sigma}\partial_\mu A^b_\nu\partial_\rho A^c_\sigma\,\mathrm{Tr}(T^aT^bT^c)$. Since $\varepsilon^{\mu\nu\rho\sigma}$ is symmetric on exchange of $\mu\nu \leftrightarrow \rho\sigma$, $\varepsilon^{\mu\nu\rho\sigma}\partial_\mu A^b_\nu\partial_\rho A^c_\sigma$ is symmetric on exchange of $b \leftrightarrow c$. Thus only the symmetric d^{abc} term in eq. (77.37) survives, and this is proportional to $A(\mathrm{R})$.

Now consider $\varepsilon^{\mu\nu\rho\sigma}A^b_\nu A^c_\rho A^d_\sigma\,\mathrm{Tr}(T^aT^bT^cT^d)$. We note that $\varepsilon^{\mu\nu\rho\sigma}A^b_\nu A^c_\rho A^d_\sigma$ is antisymmetric on $c \leftrightarrow d$. Thus we can replace T^cT^d with its antisymmetric part, $\frac{1}{2}[T^c,T^d] = \frac{1}{2}if^{cde}T^e$. Then we have $\mathrm{Tr}(T^aT^b[T^c,T^d]) = -\frac{1}{2}T(\mathrm{R})f^{cde}f^{abe} + iA(\mathrm{R})f^{cde}d^{abe}$. This must then be contracted with $\varepsilon^{\mu\nu\rho\sigma}A^b_\nu A^c_\rho A^d_\sigma$, which is completely antisymmetric on bcd. We can make $f^{cde}f^{abe}$ and $f^{cde}d^{abe}$ completely antisymmetric on bcd by adding the two cyclic permutations of bcd (and dividing by 3). For the ff term, we get $\frac{1}{3}(f^{cde}f^{abe} + f^{dbe}f^{ace} + f^{bce}f^{ade})$, and this vanishes by the Jacobi identity. There is no comparable identity for the fd term, so this does not vanish, and is proportional to $A(\mathrm{R})$.

77.2) We note that $\varepsilon^{\mu\nu\rho\sigma}\partial_\mu(A_\nu\partial_\rho A_\sigma) = \varepsilon^{\mu\nu\rho\sigma}\partial_\mu A_\nu\partial_\rho A_\sigma = \frac{1}{4}\varepsilon^{\mu\nu\rho\sigma}F^0_{\mu\nu}F^0_{\rho\sigma}$, where we have defined $F^0_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$.

Next we note that $\varepsilon^{\mu\nu\rho\sigma}\partial_\mu\mathrm{Tr}(A_\nu A_\rho A_\sigma) = 3\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}[(\partial_\mu A_\nu)A_\rho A_\sigma] = \frac{3}{4}\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}(F^0_{\mu\nu}[A_\rho,A_\sigma])$.

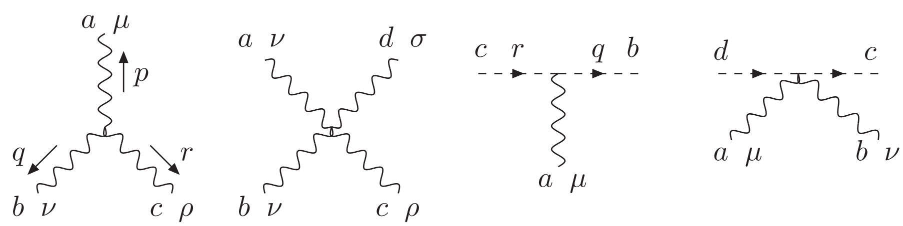
Next we note that $\mathrm{Tr}([A_\mu,A_\nu][A_\rho,A_\sigma]) = A^a_\mu A^b_\nu A^c_\rho A^d_\sigma\,\mathrm{Tr}([T^a,T^b][T^c,T^d])$. Then we have $\mathrm{Tr}([T^a,T^b][T^c,T^d]) = -f^{abe}f^{cdg}\,\mathrm{Tr}(T^eT^g) = -T(\mathrm{R})f^{abe}f^{cde}$. If we contract with $\varepsilon^{\mu\nu\rho\sigma}$, we have a factor of $\varepsilon^{\mu\nu\rho\sigma}A^a_\mu A^b_\nu A^c_\rho A^d_\sigma$, which is completely antisymmetric on abc. (Actually, on $abcd$, but abc will be enough for our purposes.) We can make $f^{abe}f^{cde}$ completely antisymmetric on abc by adding the two cyclic permutations of abc (and dividing by 3). As in the previous problem, the result vanishes by the Jacobi identity. Thus, $\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}([A_\mu,A_\nu][A_\rho,A_\sigma]) = 0$.

Putting all this together, and using $F_{\mu\nu} = F^0_{\mu\nu} - ig[A_\mu,A_\nu]$, we have

$$\begin{aligned}\tfrac{1}{4}\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}(F_{\mu\nu}F_{\rho\sigma}) &= \tfrac{1}{4}\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}((F^0_{\mu\nu}-ig[A_\mu,A_\nu])(F^0_{\rho\sigma}-ig[A_\rho,A_\sigma])) \\ &= \tfrac{1}{4}\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}(F^0_{\mu\nu}F^0_{\rho\sigma} - 2igF^0_{\mu\nu}[A_\rho,A_\sigma]) \\ &= \varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}(\partial_\mu A_\nu\partial_\rho A_\sigma - \tfrac{2}{3}ig\partial_\mu(A_\nu A_\rho A_\sigma)) \\ &= \varepsilon^{\mu\nu\rho\sigma}\partial_\mu\mathrm{Tr}(A_\nu\partial_\rho A_\sigma - \tfrac{2}{3}igA_\nu A_\rho A_\sigma)\ .\end{aligned} \tag{77.38}$$

78 Background Field Gauge

78.1) The relevant vertices are



For completeness we write down the corresponding vertex factors when all fields are internal, as given in section 72:

$$i\mathbf{V}^{abc}_{\mu\nu\rho}(p,q,r) = gf^{abc}[(q-r)_\mu g_{\nu\rho} + (r-p)_\nu g_{\rho\mu} + (p-q)_\rho g_{\mu\nu}] \,, \tag{78.43}$$

$$\begin{aligned} i\mathbf{V}^{abcd}_{\mu\nu\rho\sigma} = -ig^2\,[\, & f^{abe}f^{cde}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \\ & + f^{ace}f^{dbe}(g_{\mu\sigma}g_{\rho\nu} - g_{\mu\nu}g_{\rho\sigma}) \\ & + f^{ade}f^{bce}(g_{\mu\nu}g_{\sigma\rho} - g_{\mu\rho}g_{\sigma\nu})\,] \,, \end{aligned} \tag{78.44}$$

$$i\mathbf{V}^{abc}_{\mu}(q,r) = gf^{abc}q_\mu \,, \tag{78.45}$$

$$i\mathbf{V}^{abcd}_{\mu\nu} = 0 \,. \tag{78.46}$$

In the three-gluon vertex, at most one gluon may be external. We then have

$$i\mathbf{V}^{\bar{a}bc}_{\bar{\mu}\nu\rho}(\bar{p},q,r) = gf^{\bar{a}bc}[(q-r)_{\bar{\mu}} g_{\nu\rho} + (r-\bar{p}+q/\xi)_\nu g_{\rho\bar{\mu}} + (\bar{p}-q-r/\xi)_\rho g_{\bar{\mu}\nu}] \,, \tag{78.47}$$

where we have put bars over the labels of the external line ($\bar{p}$, $\bar{\mu}$, $\bar{a}$) to identify it. Also, we have left the gauge-fixing parameter ξ arbitrary.

In the four-gluon vertex, at most two gluons may be external. If just one is external, the vertex is unchanged. If two are external, we have

$$\begin{aligned} i\mathbf{V}^{\bar{a}\bar{b}cd}_{\bar{\mu}\bar{\nu}\rho\sigma} = -ig^2\,[\, & f^{\bar{a}\bar{b}e}f^{cde}(g_{\bar{\mu}\rho}g_{\bar{\nu}\sigma} - g_{\bar{\mu}\sigma}g_{\bar{\nu}\rho}) \\ & + f^{\bar{a}ce}f^{d\bar{b}e}(g_{\bar{\mu}\sigma}g_{\rho\bar{\nu}} - g_{\bar{\mu}\bar{\nu}}g_{\rho\sigma} - g_{\bar{\mu}\rho}g_{\bar{\nu}\sigma}/\xi) \\ & + f^{\bar{a}de}f^{\bar{b}ce}(g_{\bar{\mu}\nu}g_{\sigma\rho} - g_{\bar{\mu}\rho}g_{\sigma\nu} + g_{\bar{\mu}\sigma}g_{\bar{\nu}\rho}/\xi)\,] \,. \end{aligned} \tag{78.48}$$

In the gluon-ghost-ghost vertex, if the gluon is external we have

$$i\mathbf{V}^{\bar{a}bc}_{\bar{\mu}}(q,r) = gf^{\bar{a}bc}(q+r)_{\bar{\mu}} \,. \tag{78.49}$$

In the gluon-gluon-ghost-ghost vertex, one or both gluons may be external. If just one is external, we have

$$i\mathbf{V}^{\bar{a}bcd}_{\bar{\mu}\nu} = -ig^2 f^{\bar{a}ce}f^{bde}g_{\bar{\mu}\nu} \,. \tag{78.50}$$

If both are external, we have

$$i\mathbf{V}^{\bar{a}\bar{b}cd}_{\bar{\mu}\bar{\nu}} = -ig^2(f^{\bar{a}ce}f^{\bar{b}de} + f^{\bar{b}ce}f^{\bar{a}de})g_{\bar{\mu}\bar{\nu}} \,. \tag{78.51}$$

See L. F. Abbott, Nucl. Phys. B185, 189 (1981) for more details (and a two-loop calculation of the beta function).

78.2) The general analysis of section 53 shows that if we integrate out a field ϕ with a lagrangian of the form $\phi\Box\phi$, we get $(\det\Box)^\nu$, where ν is negative for a bosonic field and positive for a fermionic field, with magntiude $|\nu| = \frac{1}{2}$ for a real field and $|\nu| = 1$ for a complex field (or a pair of real fields). We can then compute $(\det\Box)^\nu$ by summing all diagrams with a single ϕ loop. (If the lagrangian for ϕ includes cubic or higher terms, these generate vertices that enter only at higher-loop order.) Thus to verify eq. (78.42) at the one-loop level, we need only show that the quadratic terms for the fields we integrate out (namely $\mathcal{A}$, c, $\bar{c}$, and Ψ) have the form $\phi\Box\phi$ with the appropriate $\Box$.

For the ghost fields, we see from eq. (78.27) that the quadratic term can be written as $\bar{c}^b(\bar{D}^2)^{bc}c^c$, with $\bar{D}^\mu$ in the adjoint representation. From eqs. (78.38) and (78.39), we see that $\bar{D}^2 = \Box_{\mathrm{A},(1,1)}$.

For the quantum gauge field, we use $f^{abc} = i(T^a_{\mathrm{A}})^{bc}$ to write the last term in eq. (78.27) (omitting the Z_3) as $-ig\mathcal{A}^{b\alpha}[(T^a_{\mathrm{A}})^{bc}\bar{F}^a_{\alpha\beta}]\mathcal{A}^{c\beta} = \frac{1}{2}g\mathcal{A}^{b\alpha}[(T^a_{\mathrm{A}})^{bc}\bar{F}^a_{\mu\nu}(S^{\mu\nu}_{(2,2)})_{\alpha\beta}]\mathcal{A}^{c\beta}$. The complete quadratic term for $\mathcal{A}$ is then $\frac{1}{2}\mathcal{A}^{b\alpha}[(\bar{D}^2)^{bc}g_{\alpha\beta} + g(T^a_{\mathrm{A}})^{bc}\bar{F}^a_{\mu\nu}(S^{\mu\nu}_{(2,2)})_{\alpha\beta}]\mathcal{A}^{c\beta}$. From eqs. (78.38) and (78.39), we see that the operator in square brackets is $\Box_{\mathrm{A},(2,2)}$.

For a massless Dirac fermion, the lagrangian is $\overline{\Psi}(i\bar{\not{D}})\Psi$, and so integrating it out yields $\det(i\bar{\not{D}}) = [\det(i\bar{\not{D}})^2]^{1/2}$. From eq. (77.27) with $k = 0$, we have $(i\bar{\not{D}})^2 = \bar{D}^2 + gS^{\mu\nu}\bar{F}_{\mu\nu}$, where $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ and $\bar{F}_{\mu\nu} = T^a_{\mathrm{R}}\bar{F}^a_{\mu\nu}$. From eqs. (78.38) and (78.39), we see that $(i\bar{\not{D}})^2 = \Box_{\mathrm{R_{DF}},(2,1)\oplus(1,2)}$.

For more details, see *Peskin & Schroeder* or *Weinberg II*.

79 Gervais–Neveu Gauge

80 The Feynman Rules for $N \times N$ Matrix Fields

80.1) For $N = 1$, every $T^a = 1$, and so in terms of component fields (actually of course just the one field), the cubic vertex factor is $2ig$ and the quartic vertex factor is $-6i\lambda$. Diagrams contributing to $\varphi\varphi \to \varphi\varphi$ scattering are those of fig. 10.2, plus a four-point vertex; thus the amplitude is

$$\mathcal{T} = \frac{(2g)^2}{(k_1+k_2)^2} + \frac{(2g)^2}{(k_1+k_3)^2} + \frac{(2g)^2}{(k_1+k_4)^2} - 6\lambda \,. \tag{80.20}$$

In terms of the color-ordered rules, every trace equals one, and so summing over the six color orderings yields $2A_2 + 2A_3 + 2A_4$, where A_i is given by eqs. (80.13–15). This reproduces eq. (80.20).

80.2) The square of any trace is given by the left side of fig. 80.4, but with the right half labeled 1234 (from top to bottom). The product obviously contains four closed loops, and hence equals N^4. For the product of two different orderings (with the left half ordered as 1234 by convention), we can always use the cyclic property of the trace to put the label 1 at the top of the right half. Then any of the five possible ordering of 234 (that are not equal to 234) differ either by exchange of a single pair (324, 243, 432) or by a cyclic permutation (342, 423). In either case it is easy to see that the product contains two closed loops, and hence equals N^2.

80.3) The second term in fig. 80.5 can appear on $n = 0, 1, 2, 3$, or all 4 of the bridges connecting the left half of the left side of fig. 80.4 with its mirror image; call each such appearance a "broken bridge". The number of ways to choose the n broken bridges is $C_{4,n} = 4!/(4-n)!n!$. The number of closed loops when there are n broken bridges is B_n, where $B_n = 4-n$ for $n \neq 4$, and $B_4 = 2$. Each broken bridge contributes a factor of $-1/N$. Thus, we have

$$\begin{aligned} \sum_{a_1,a_2,a_3,a_4} |\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})|^2 &= \sum_{n=0}^{4} C_{4,n} N^{B_n} (-1/N)^n \\ &= N^4 - 4N^2 + 6 - 3N^{-2} \,. \end{aligned} \tag{80.21}$$

80.4) a) We get a factor of g for every 3-point vertex and a factor of $\lambda = cg^2$ for every 4-point vertex. Each face gives a closed index loop, and hence a factor of N. Thus we get $g^{V_3+2V_4} N^F$.

b) The total number of propagator endpoints is $2E$. In a vacuum diagram, every propagator endpoint is attached to a vertex. Since each n-point vertex accounts for n endpoints, we have $2E = 3V_3 + 4V_4$.

c) $\chi = V - E + F$. (See http://www.ics.uci.edu/∼eppstein/junkyard/euler for 19 different proofs in the case of $\mathcal{G} = 0$.) Using $V = V_3 + V_4$ and $2E = 3V_3 + 4V_4$, we also can write $\chi = F - \frac{1}{2}V_3 - V_4$.

d) Setting $g = \bar\lambda^{1/2} N^{-1/2}$, our result in part (a) is that the N dependence of each vacuum diagram is given by $N^{-V_3/2-V_4} N^F = N^{F-V_3/2-V_4} = N^\chi = N^{2-2\mathcal{G}}$.

81 Scattering in Quantum Chromodynamics

81.1) If we choose $q_1 = q_3 = k_2$ and $q_2 = q_4 = k_3$, then all polarization products vanish except $\varepsilon_1 \cdot \varepsilon_4 = \langle 3\,1\rangle\,[4\,2]/\langle 3\,4\rangle\,[2\,1]$. Also, the analysis beginning after eq. (81.14) and leading to eq. (81.21) still holds, and we still have $k_5 \cdot \varepsilon_2 = -k_1 \cdot \varepsilon_2$. The remaining factors are now $k_1 \cdot \varepsilon_2 = \langle 3\,1\rangle\,[1\,2]/\sqrt{2}\,\langle 3\,2\rangle$ and $k_4 \cdot \varepsilon_3 = [2\,4]\,\langle 4\,3\rangle\,/\sqrt{2}\,[2\,3]$. Putting all this together and canceling (most) common factors, we find

$$A = \frac{\langle 1\,3\rangle^2\,[2\,4]^2\,\langle 3\,4\rangle}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,[1\,2]\,[2\,3]}\;. \tag{81.61}$$

In the numerator, use $[2\,4]\,\langle 3\,4\rangle = -[2\,1]\,\langle 3\,1\rangle$, and cancel $-[2\,1]$ with the $[1\,2]$ in the denominator. Now multiply numerator and denominator by $\langle 4\,1\rangle$, use $[2\,4]\,\langle 4\,1\rangle = -[2\,3]\,\langle 3\,1\rangle$ in the numerator, and cancel the $[2\,3]$ with the one in the denominator. Using $-\langle 3\,1\rangle = \langle 1\,3\rangle$ in the numerator then yields

$$A = \frac{\langle 1\,3\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,1\rangle}\;, \tag{81.62}$$

which agrees with eq. (81.37).

81.2) Using $p_5 = -p_1 - k_4$ in the first diagram, we see that the first line of eq. (81.47) is proportional to $[2|\not{\varepsilon}_3\not{p}_1\not{\varepsilon}_4|1\rangle + [2|\not{\varepsilon}_3\not{k}_4\not{\varepsilon}_4|1\rangle$. With $q_4 = k_1$, we have $\not{\varepsilon}_4 \propto |4\rangle[1| + |1]\langle 4|$, and so $\not{\varepsilon}_4|1\rangle \propto |1]\langle 4\,1\rangle$. We also have $\not{p}_1 \propto |1\rangle[1| + |1]\langle 1|$, and so $\not{p}_1\not{\varepsilon}_4|1\rangle = 0$. Similarly, with $q_3 = k_4$, we have $\not{\varepsilon}_3 \propto |3]\langle 4| + |4\rangle[3|$, and so $[2|\not{\varepsilon}_3 \propto [2\,3]\langle 4|$. We also have $\not{k}_4 \propto |4\rangle[4| + |4]\langle 4|$, and so $[2|\not{\varepsilon}_3\not{k}_4 = 0$. Thus both terms in the first line of eq. (81.47) vanish with this choice of reference momenta.

In the second diagram, we have $\mathbf{V}_{345} = -i\sqrt{2}g[(\varepsilon_3\varepsilon_4)(k_3\varepsilon_5) + (\varepsilon_4\varepsilon_5)(k_4\varepsilon_3) + (\varepsilon_5\varepsilon_3)(k_5\varepsilon_4)]$ with $k_5 = -k_3 - k_4$. For $q_3 = k_4$, $\varepsilon_3 \cdot \varepsilon_4 = 0$ and $k_4 \cdot \varepsilon_3 = 0$, so the first two terms in $\mathbf{V}_{345}$ vanish. Also, $k_5 \cdot \varepsilon_4 = -k_3 \cdot \varepsilon_4 - k_4 \cdot \varepsilon_4 = -k_3 \cdot \varepsilon_4$. Thus $\mathbf{V}_{345} = i\sqrt{2}g(\varepsilon_5\varepsilon_3)(k_3\varepsilon_4)$. Making the replacement $\varepsilon_5^\mu \varepsilon_5^\nu \to ig^{\mu\nu}/s_{12}$ then yields

$$A = -\,\frac{[2|\not{\varepsilon}_3|1\rangle\,k_3 \cdot \varepsilon_4}{s_{12}}\;. \tag{81.63}$$

We have $[2|\not{\varepsilon}_3|1\rangle = \sqrt{2}\,[2\,3]\,\langle 4\,1\rangle/\langle 4\,3\rangle$, $k_3 \cdot \varepsilon_4 = [1\,3]\,\langle 3\,4\rangle/\sqrt{2}\,[1\,4]$, and $s_{12} = \langle 1\,2\rangle\,[2\,1]$; therefore

$$A = -\,\frac{[2\,3]\,\langle 4\,1\rangle\,[1\,3]\,\langle 3\,4\rangle}{\langle 4\,3\rangle\,[1\,4]\,\langle 1\,2\rangle\,[2\,1]}\;. \tag{81.64}$$

In the numerator, use $[1\,3]\,\langle 3\,4\rangle = -[1\,2]\,\langle 2\,4\rangle$, and cancel the $-[1\,2]$ with $[2\,1]$ in the denominator. Now multiply numerator and denominator by $\langle 4\,1\rangle$, use $[1\,4]\,\langle 4\,1\rangle = s_{14} = s_{23} = \langle 2\,3\rangle\,[3\,2]$ in the denominator, and cancel the $[3\,2]$ with $-[2\,3]$ in the numerator. Finally, multiply numerator and denominator by $\langle 4\,1\rangle$ and use $\langle 4\,1\rangle = -\langle 1\,4\rangle$ to get

$$A = \frac{\langle 1\,4\rangle^3\,\langle 2\,4\rangle}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,1\rangle}\;, \tag{81.65}$$

which agrees with eq. (81.55).

81.3) We take $q_3 = k_4$ and $q_4 = k_3$ and follow the analysis that led to eq. (81.50), which now reads

$$A = \tfrac{1}{2}\,[2|\not\varepsilon_{3-}(\not p_1+\not k_4)\not\varepsilon_{4+}|1\rangle/(-s_{14})\ . \tag{81.66}$$

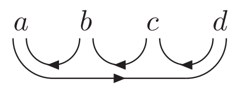
We have $\not\varepsilon_{3-} = \sqrt{2}(|3\rangle[4| + |4]\langle 3|)/[4\,3]$, $\not\varepsilon_{4+} = \sqrt{2}(|3\rangle[4| + |4]\langle 3|)/\langle 3\,4\rangle$, so

$$A = \frac{[2\,4]\,\langle 3\,1\rangle\,[1\,4]\,\langle 3\,1\rangle}{[4\,3]\,\langle 3\,4\rangle\,s_{14}}\ . \tag{81.67}$$

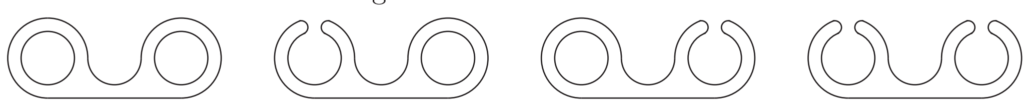
Use $s_{14} = \langle 4\,1\rangle\,[1\,4]$ and cancel the $[1\,4]$. Now multiply numerator and denominator by $\langle 1\,3\rangle$ and use $\langle 1\,3\rangle\,[4\,3] = -\langle 1\,2\rangle\,[4\,2]$ in the denominator, and cancel the $-[4\,2]$ with $[2\,4]$. Finally multiply numerator and denominator by $\langle 2\,3\rangle$ to get

$$A = \frac{\langle 1\,3\rangle^3\,\langle 2\,3\rangle}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,1\rangle}\ , \tag{81.68}$$

which is the same as eq. (81.56).

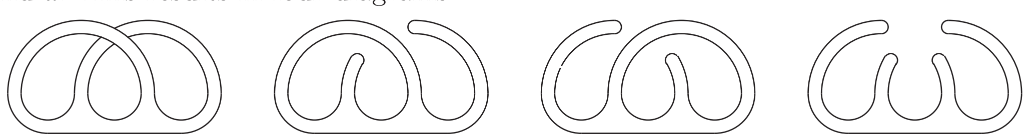
81.4) a) The double-line picture for $\mathrm{Tr}(T^aT^bT^cT^d)$ is 

To compute $\mathrm{Tr}(T^aT^aT^cT^c)$, we connect a and b with the propagator of fig. 80.5, and also c and d. This results in four diagrams



with coefficients 1, $-1/N$, $-1/N$, and $+1/N^2$; arrows have been omitted. Each closed loop results in a factor of N; these diagrams have 3, 2, 2, and 1 closed loops, respectively. Thus, $\mathrm{Tr}(T^aT^aT^cT^c) = N^3 - (1/N)N^2 - (1/N)N^2 + (1/N^2)N = (N^2{-}1)^2/N$.

To compute $\mathrm{Tr}(T^aT^bT^aT^b)$, we connect a and c with the propagator of fig. 80.5, and also b and d. This results in four diagrams



with coefficients 1, $-1/N$, $-1/N$, and $+1/N^2$; arrows have been omitted. Each closed loop results in a factor of N; these diagrams have 1, 2, 2, and 1 closed loops, respectively. Thus, $\mathrm{Tr}(T^aT^aT^cT^c) = N - (1/N)N^2 - (1/N)N^2 + (1/N^2)N = -(N^2{-}1)/N$.

b) Using the cyclic property of the trace, we have $\mathrm{Tr}(T^a_{\rm R}T^b_{\rm R}T^b_{\rm R}T^a_{\rm R}) = \mathrm{Tr}(T^a_{\rm R}T^a_{\rm R}T^b_{\rm R}T^b_{\rm R})$. Using $T^a_{\rm R}T^a_{\rm R} = C({\rm R})I$, we get $\mathrm{Tr}(T^a_{\rm R}T^b_{\rm R}T^b_{\rm R}T^a_{\rm R}) = C^2({\rm R})D({\rm R}) = T^2({\rm R})D^2({\rm A})/D({\rm R})$.

We use $T^a_{\rm R}T^b_{\rm R} = T^b_{\rm R}T^a_{\rm R} + if^{abc}T^c_{\rm R}$ to get

$$\begin{aligned}
\mathrm{Tr}(T^a_{\rm R}T^b_{\rm R}T^a_{\rm R}T^b_{\rm R}) &= \mathrm{Tr}(T^b_{\rm R}T^a_{\rm R}T^a_{\rm R}T^b_{\rm R}) + if^{abc}\,\mathrm{Tr}(T^c_{\rm R}T^a_{\rm R}T^b_{\rm R}) \\
&= T^2({\rm R})D^2({\rm A})/D({\rm R}) + \tfrac{1}{2}if^{abc}\,\mathrm{Tr}(T^c_{\rm R}[T^a_{\rm R},T^b_{\rm R}]) \\
&= T^2({\rm R})D^2({\rm A})/D({\rm R}) - \tfrac{1}{2}f^{abc}f^{abd}\,\mathrm{Tr}(T^c_{\rm R}T^d_{\rm R}) \\
&= T^2({\rm R})D^2({\rm A})/D({\rm R}) - \tfrac{1}{2}T({\rm A})\delta^{cd}\,T({\rm R})\delta^{cd} \\
&= T^2({\rm R})D^2({\rm A})/D({\rm R}) - \tfrac{1}{2}T({\rm A})T({\rm R})D({\rm A})\ .
\end{aligned} \tag{81.69}$$

Using $T({\rm N}) = 1$, $T({\rm A}) = 2N$, $D({\rm R}) = N$, and $D({\rm A}) = N^2{-}1$, we reproduce eqs. (81.58) and (81.59).

81.5) Squaring eqs. (81.55) and (81.56), we have

$$|A(1^-_{\bar q},2^+_q,3^+,4^-)|^2 = s^3_{14}s_{13}/s^2_{12}s^2_{14} = s_{14}s_{13}/s^2_{12} = ut/s^2 \ ,$$
$$|A(1^-_{\bar q},2^+_q,3^-,4^+)|^2 = s^3_{13}s_{14}/s^2_{12}s^2_{14} = s^3_{13}/s^2_{12}s_{14} = t^3/s^2u \ . \tag{81.70}$$

Changing the signs of all helicities is equivalent to complex conjugation, and thus yields the same values of $|A|^2$. Thus we have

$$\sum_{\text{helicities}} |A_3|^2 = 2(t^3/u + tu)/s^2 \ . \tag{81.71}$$

Swapping $3 \leftrightarrow 4$ is equivalent to $t \leftrightarrow u$, so

$$\sum_{\text{helicities}} |A_4|^2 = 2(u^3/t + tu)/s^2 \ . \tag{81.72}$$

Adding these, we find

$$\sum_{\text{helicities}} \Big(|A_3|^2 + |A_4|^2\Big) = 2(t^3/u + 2tu + u^3/t)/s^2 \ . \tag{81.73}$$

We must also evaluate $A^*(1^-_{\bar q},2^+_q,3^+,4^-)A(1^-_{\bar q},2^+_q,4^-,3^+)$. Complex conjugation changes all angle brackets to square brackets, with an even number of additional minus signs. Then we have

$$A^*(1^-_{\bar q},2^+_q,3^+,4^-)A(1^-_{\bar q},2^+_q,4^-,3^+) = \frac{[1\,4]^3\,[2\,4]\,\langle 1\,4\rangle^3\,\langle 2\,4\rangle}{[1\,2]\,[2\,3]\,[3\,4]\,[4\,1]\,\langle 1\,2\rangle\langle 2\,4\rangle\langle 4\,3\rangle\langle 3\,1\rangle}$$
$$= \frac{(-s_{14})^3\,(-s_{24})}{(-s_{12})\,(s_{34})\,[2\,3]\,\langle 3\,1\rangle\,[4\,1]\,\langle 2\,4\rangle} \ . \tag{81.74}$$

Now we use $[2\,3]\,\langle 3\,1\rangle = -[2\,4]\,\langle 4\,1\rangle$ in the denominator, followed by $-[2\,4]\,\langle 2\,4\rangle = s_{24} = t$ and $-\langle 4\,1\rangle\,[4\,1] = s_{14} = u$ and $s_{34} = s_{12} = s$ to get

$$A^*(1^-_{\bar q},2^+_q,3^+,4^-)A(1^-_{\bar q},2^+_q,4^-,3^+) = u^2/s^2 \ . \tag{81.75}$$

This is real, so taking the complex conjugate yields

$$A^*(1^-_{\bar q},2^+_q,4^-,3^+)A(1^-_{\bar q},2^+_q,3^+,4^-) = u^2/s^2 \ . \tag{81.76}$$

Swapping $3 \leftrightarrow 4$ in eqs. (81.75) and (81.76) yields

$$A^*(1^-_{\bar q},2^+_q,4^+,3^-)A(1^-_{\bar q},2^+_q,3^-,4^+) = t^2/s^2 \ , \tag{81.77}$$

$$A^*(1^-_{\bar q},2^+_q,3^-,4^+)A(1^-_{\bar q},2^+_q,4^+,3^-) = t^2/s^2 \ . \tag{81.78}$$

Adding up eqs. (81.75–81.78) and the same with all helicities flipped (which is equivalent to complex conjugation), we find

$$\sum_{\text{helicities}} \Big(A^*_3 A_4 + A^*_4 A_3\Big) = 4(t^2 + u^2)/s^2 \ . \tag{81.79}$$

81.6) See page 19 of "Calculating Scattering Amplitudes Efficiently" by Lance Dixon, available online at http://arXiv.org/hep-ph/9601359.

82 Wilson Loops, Lattice Theory, and Confinement

82.1) We have

$$\langle 0|W_C|0\rangle = \exp\left[-\frac{g^2}{8\pi^2}\oint_C dx_\mu \oint_C dy_\mu \,\frac{1}{(x-y)^2}\right], \tag{82.42}$$

where C is a circle of radius R. Since the integrand depends only on $x-y$, we can fix $y_\mu = R(1,0)$ and replace $\oint_C dy_\mu$ with $2\pi R(0,1)$. Then we set $x_\mu = R(\cos\theta,\sin\theta)$ and $dx_\mu = R(-\sin\theta,\cos\theta)d\theta$. We then have

$$\begin{aligned}\oint_C dx_\mu \oint_C dy_\mu \,\frac{1}{(x-y)^2} &= 2\pi\int_{-\pi}^{+\pi}\frac{\cos\theta\,d\theta}{(\cos\theta-1)^2+(\sin\theta)^2}\\ &= 2\pi\int_{-\pi}^{+\pi}\frac{\cos\theta\,d\theta}{2(1-\cos\theta)}\\ &= 2\pi\int_0^{\pi}\frac{\cos\theta\,d\theta}{1-\cos\theta}\;.\end{aligned} \tag{82.43}$$

We are instructed to set the integrand to zero if $(x-y)^2 < a^2$; since $(x-y)^2 = 2R^2(1-\cos\theta) \simeq R^2\theta^2$ for $\theta \ll 1$, the lower limit of integration should be a/R rather than zero. The integral is then dominated by the low end, and we can make the replacements $\cos\theta \to 1$ in the numerator, and $1-\cos\theta \to \frac{1}{2}\theta^2$ in the denominator. Then we have

$$\begin{aligned}\oint_C dx_\mu \oint_C dy_\mu \,\frac{1}{(x-y)^2} &= 4\pi\int_{a/R}\frac{d\theta}{\theta^2}\\ &= \frac{4\pi R}{a} + O(1)\\ &= \frac{2P}{a} + O(1)\;.\end{aligned} \tag{82.44}$$

Thus

$$\langle 0|W_C|0\rangle = \exp[-(g^2/4\pi^2 a)P]\;, \tag{82.45}$$

so $\tilde{c} = 1/4\pi^2$.

83 Chiral Symmetry Breaking

83.1) a) Each Dirac field equals two left-handed Weyl fields. All $2n_{\rm F}$ of these Weyl fields are in the 3 representation (because it is real). So there is a $\mathrm{U}(2n_{\rm F})$ flavor symmetry; the U(1) is anomalous, leaving a nonanomalous flavor symmetry group $\mathrm{SU}(2n_{\rm F})$.

b) Call the Weyl fields $\chi_{\alpha i}$, $\alpha = 1,2,3$, $i = 1,\ldots,2n_{\rm F}$. The composite field is $\chi_{\alpha i}\chi_{\alpha j}$, and it is symmetric on $i \leftrightarrow j$. The condensate is $\langle 0|\chi_{\alpha i}\chi_{\alpha j}|0\rangle = -v^3\delta_{ij}$. The general $\mathrm{SU}(2n_{\rm F})$ flavor transformation is $\chi_{\alpha i} \to L_{ij}\chi_{\alpha j}$. The δ_{ij} in the condensate transforms to $L_{ik}L_{jk} = (LL^{\rm T})_{ij}$. For this to equal δ_{ij}, L must be orthogonal. Thus the unbroken flavor symmetry group is $\mathrm{SO}(2n_{\rm F})$.

c) Number of Goldstone bosons = number of generators of SU(4) minus the number of generators of O(4) = 15 − 6 = 9.

d) The nonanomalous flavor symmetry is again $\mathrm{SU}(2n_{\rm F})$. The composite field is $\varepsilon^{\alpha\beta}\chi_{\alpha i}\chi_{\beta j}$, and it is *antisymmetric* on $i \leftrightarrow j$. The condensate is $\langle 0|\varepsilon^{\alpha\beta}\chi_{\alpha i}\chi_{\beta j}|0\rangle = -v^3\eta_{ij}$, where $\eta_{ij} = -\eta_{ji}$. We assume that $\eta^2 = -I$, which yields the largest possible unbroken subgroup, $\mathrm{Sp}(2n_{\rm F})$; see problem 24.4. Number of Goldstone bosons = number of generators of SU(4) minus the number of generators of Sp(4) = 15 − 10 = 5.

83.2) So that $\langle 0|\mathcal{H}_{\rm mass}|0\rangle$ is negative, and lowers the energy.

83.3) Let $\Pi(x) = \pi^a(x)T^a/f_\pi$. Then $U = 1 + 2i\Pi - 2\Pi^2 - \frac{4}{3}i\Pi^3$ and $\partial_\mu U = 2i\partial_\mu\Pi - 2[(\partial_\mu\Pi)\Pi + \Pi(\partial_\mu\Pi)] - \frac{4}{3}i[(\partial_\mu\Pi)\Pi^2 + \Pi(\partial_\mu\Pi)\Pi + \Pi^2(\partial_\mu\Pi)]$. $\partial_\mu U^\dagger$ is the same, with $i \to -i$. Then

$$\begin{aligned}\partial^\mu U^\dagger\partial_\mu U &= 4\partial^\mu\Pi\partial_\mu\Pi + 4[(\partial_\mu\Pi)\Pi + \Pi(\partial_\mu\Pi)][(\partial_\mu\Pi)\Pi + \Pi(\partial_\mu\Pi)] \\ &\quad - \tfrac{8}{3}\partial_\mu\Pi[(\partial_\mu\Pi)\Pi^2 + \Pi(\partial_\mu\Pi)\Pi + \Pi^2(\partial_\mu\Pi)] \\ &\quad - \tfrac{8}{3}[(\partial_\mu\Pi)\Pi^2 + \Pi(\partial_\mu\Pi)\Pi + \Pi^2(\partial_\mu\Pi)]\partial_\mu\Pi\,. \end{aligned} \tag{83.35}$$

Taking the trace and using the cyclic property, we find

$$\begin{aligned}\mathrm{Tr}\,\partial^\mu U^\dagger\partial_\mu U &= 4\,\mathrm{Tr}\,\partial^\mu\Pi\partial_\mu\Pi + (8-\tfrac{32}{3})\mathrm{Tr}\,\Pi^2\partial^\mu\Pi\partial_\mu\Pi + (8-\tfrac{16}{3})\mathrm{Tr}\,\Pi(\partial^\mu\Pi)\Pi(\partial_\mu\Pi) \\ &= 4f_\pi^{-2}\pi^a\pi^b\,\mathrm{Tr}\,T^aT^b - \tfrac{8}{3}f_\pi^{-4}[\pi^a\pi^b\partial^\mu\pi^c\partial_\mu\pi^d - \pi^a(\partial^\mu\pi^b)\pi^c(\partial_\mu\pi^d)]\mathrm{Tr}\,T^aT^bT^cT^d\,. \end{aligned} \tag{83.36}$$

$\mathrm{Tr}\,T^aT^b = \frac{1}{2}\delta^{ab}$. For SU(2), $\mathrm{Tr}\,T^aT^bT^cT^d$ vanishes unless the indices match in pairs. Then, using $T^aT^b = -T^bT^a$ if $a \neq b$ and $(T^a)^2 = \frac{1}{4}I$, we get $\mathrm{Tr}\,T^aT^bT^cT^d = \frac{1}{8}(\delta^{ab}\delta^{cd} - \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc})$. Using this in eq. (83.36) yields eq. (83.13).

83.4) We need the interactions from the mass term,

$$\begin{aligned}\mathcal{L}_{\rm mass} &= mv^3\,\mathrm{Tr}(U + U^\dagger) \\ &= mv^3\,\mathrm{Tr}(2 - 4\Pi^2 + \tfrac{4}{3}\Pi^4) \\ &= -4mv^3f_\pi^{-2}\pi^a\pi^b\,\mathrm{Tr}\,T^aT^b + \tfrac{4}{3}mv^3f_\pi^{-4}\pi^a\pi^b\pi^c\pi^d\,\mathrm{Tr}\,T^aT^bT^cT^d \\ &= -2mv^3f_\pi^{-2}\pi^a\pi^a + \tfrac{1}{6}mv^3f_\pi^{-4}\pi^a\pi^a\pi^b\pi^b \\ &= -\tfrac{1}{2}m_\pi^2\pi^a\pi^a + \tfrac{1}{24}m_\pi^2f_\pi^{-2}\pi^a\pi^a\pi^b\pi^b\,. \end{aligned} \tag{83.37}$$

Combing the interaction terms in eq. (83.13) and eq. (83.37), we have

$$\mathcal{L}_{\rm int} = \tfrac{1}{6}f_\pi^{-2}(\pi^a\pi^a\partial^\mu\pi^b\partial_\mu\pi^b - \pi^a\pi^b\partial^\mu\pi^b\partial_\mu\pi^a + \tfrac{1}{4}m_\pi^2\pi^a\pi^a\pi^b\pi^b) \,. \tag{83.38}$$

Treat all momenta as outgoing. Then

$$\begin{aligned}\mathcal{T} &= \mathbf{V}^{abcd}(k_a,k_b,k_c,k_d) \\ &= \tfrac{1}{3}f_\pi^{-2}[\,\delta^{ab}\delta^{cd}(-2k_ak_b - 2k_ck_d + k_ak_c + k_ak_d + k_bk_c + k_bk_d + m_\pi^2) + (bcd \to cdb, dbc)\,] \\ &= f_\pi^{-2}[\,\delta^{ab}\delta^{cd}(s-m_\pi^2) + \delta^{ac}\delta^{db}(t-m_\pi^2) + \delta^{ad}\delta^{bc}(u-m_\pi^2)\,] \,. \end{aligned} \tag{83.39}$$

83.5) Let $N_{\rm L} \equiv P_{\rm L}N$, $N_{\rm R} \equiv P_{\rm R}N$, and similarly for $\mathcal{N}$. Then

$$\begin{aligned}\mathcal{L} &= i\overline{N}_{\rm L}\partial\!\!\!/ N_{\rm L} + i\overline{N}_{\rm R}\partial\!\!\!/ N_{\rm R} - m_N(\overline{N}_{\rm R}U^\dagger N_{\rm L} + \overline{N}_{\rm L}UN_{\rm R}) \\ &\quad - \tfrac{1}{2}(g_{\rm A}{-}1)i[\,\overline{N}_{\rm L}U(\partial\!\!\!/ U^\dagger)N_{\rm L} + \overline{N}_{\rm R}U^\dagger(\partial\!\!\!/ U)N_{\rm R}] \\ &= i\overline{\mathcal{N}}\partial\!\!\!/\mathcal{N} + i\overline{\mathcal{N}}_{\rm L}(u^\dagger\partial\!\!\!/ u)\mathcal{N}_{\rm L} + i\overline{\mathcal{N}}_{\rm R}(u\partial\!\!\!/ u^\dagger)\mathcal{N}_{\rm R} - m_N\overline{\mathcal{N}}\mathcal{N} \\ &\quad - \tfrac{1}{2}(g_{\rm A}{-}1)i[\,\overline{\mathcal{N}}_{\rm L}u(\partial\!\!\!/ U^\dagger)u\mathcal{N}_{\rm L} + \overline{\mathcal{N}}_{\rm R}u^\dagger(\partial\!\!\!/ U)u^\dagger\mathcal{N}_{\rm R}] \,. \end{aligned} \tag{83.40}$$

$\partial\!\!\!/ U = (\partial\!\!\!/ u)u + u(\partial\!\!\!/ u) \Rightarrow u^\dagger(\partial\!\!\!/ U)u^\dagger = u^\dagger(\partial\!\!\!/ u) + (\partial\!\!\!/ u)u^\dagger = u^\dagger(\partial\!\!\!/ u) - u(\partial\!\!\!/ u^\dagger) \equiv -2i a\!\!\!/$. Similarly, $u(\partial\!\!\!/ U^\dagger)u = +2ia\!\!\!/$. Also, let $u^\dagger(\partial\!\!\!/ u) + u(\partial\!\!\!/ u^\dagger) \equiv -2iv\!\!\!/$. Then eq. (83.40) becomes

$$\begin{aligned}\mathcal{L} &= i\overline{\mathcal{N}}\partial\!\!\!/\mathcal{N} - m_N\overline{\mathcal{N}}\mathcal{N} + \overline{\mathcal{N}}_{\rm L}(v\!\!\!/ + a\!\!\!/)\mathcal{N}_{\rm L} + \overline{\mathcal{N}}_{\rm R}(v\!\!\!/ - a\!\!\!/)\mathcal{N}_{\rm R} \\ &\quad + (g_{\rm A}{-}1)[\,\overline{\mathcal{N}}_{\rm L}a\!\!\!/\mathcal{N}_{\rm L} - \overline{\mathcal{N}}_{\rm R}a\!\!\!/\mathcal{N}_{\rm R}\,] \\ &= i\overline{\mathcal{N}}\partial\!\!\!/\mathcal{N} - m_N\overline{\mathcal{N}}\mathcal{N} + \overline{\mathcal{N}}v\!\!\!/(P_{\rm L}{+}P_{\rm R})\mathcal{N} + g_{\rm A}\overline{\mathcal{N}}a\!\!\!/(P_{\rm L}{-}P_{\rm R})\mathcal{N} \\ &= i\overline{\mathcal{N}}\partial\!\!\!/\mathcal{N} - m_N\overline{\mathcal{N}}\mathcal{N} + \overline{\mathcal{N}}v\!\!\!/\mathcal{N} - g_{\rm A}\overline{\mathcal{N}}a\!\!\!/\gamma_5\mathcal{N} \,. \end{aligned} \tag{83.41}$$

83.6) From eq. (83.19), we find

$$\begin{aligned}\mathcal{L}_{\rm mass} &= -v^3f_\pi^{-2}\,{\rm Tr}\,M\Pi^2 \\ &= -2v^3f_\pi^{-2}[(m_u+m_d)\pi^+\pi^- + (m_u+m_s)K^+K^- + (m_d+m_s)\overline{K}^0K^0 \\ &\qquad + \tfrac{1}{2}m_u(\tfrac{1}{\sqrt3}\eta+\pi^0)^2 + \tfrac{1}{2}m_d(\tfrac{1}{\sqrt3}\eta-\pi^0)^2 + \tfrac{2}{3}m_s\eta^2] \,, \end{aligned} \tag{83.42}$$

Thus

$$m_{\pi^\pm}^2 = 2v^3f_\pi^{-2}(m_u+m_d) \tag{83.43}$$

$$m_{K^\pm}^2 = 2v^3f_\pi^{-2}(m_u+m_s) \tag{83.44}$$

$$m_{K^0\overline{K}^0}^2 = 2v^3f_\pi^{-2}(m_d+m_s) \tag{83.45}$$

$$\begin{aligned} m_{\pi^0,\eta}^2 &= \tfrac{4}{3}v^3f_\pi^{-2}[m_u+m_d+m_s \\ &\qquad \mp (m_u^2+m_d^2+m_s^2-m_um_d-m_dm_s-m_sm_u)^{1/2}] \,. \end{aligned} \tag{83.46}$$

b) Expanding in $m_{u,d}/m_s$, we find

$$m_{\pi^0}^2 = 2v^3f_\pi^{-2}(m_u+m_d) \,, \tag{83.47}$$

$$m_\eta^2 = \tfrac{2}{3}v^3f_\pi^{-2}(4m_s+m_u+m_d) \,. \tag{83.48}$$

We add $\Delta m^2_{\rm EM}$ to $m^2_{\pi^\pm}$ and $2\Delta m^2_{\rm EM}$ to $m^2_{K^\pm}$. Then we find

$$\begin{aligned}
\Delta m^2_{\rm EM} &= m^2_{\pi^\pm} - m^2_{\pi^0} = 0.00138\,{\rm GeV}^2\ , && (83.49)\\
m_u v^3 f_\pi^{-2} &= \tfrac14(+m^2_{K^\pm} - m^2_{K^0} + m^2_{\pi^0} - 2\Delta m^2_{\rm EM}) = 0.00288\,{\rm GeV}^2\ , && (83.50)\\
m_d v^3 f_\pi^{-2} &= \tfrac14(-m^2_{K^\pm} + m^2_{K^0} + m^2_{\pi^0} + 2\Delta m^2_{\rm EM}) = 0.00624\,{\rm GeV}^2\ , && (83.51)\\
m_s v^3 f_\pi^{-2} &= \tfrac14(+m^2_{K^\pm} + m^2_{K^0} - m^2_{\pi 0} - 2\Delta m^2_{\rm EM}) = 0.11777\,{\rm GeV}^2\ . && (83.52)
\end{aligned}$$

c) $m_u/m_d = 0.46$ and $m_s/m_d = 19$.

d) Using eqs. (83.50–83.52) in (83.48), we find $m_\eta = 0.566\,$GeV, 3% larger than its observed value, 0.548 GeV.

83.7) a) Focusing on the π^9 dependence, we have $U = 1 + i\pi^9/f_9$, $\det U = 1 + 3i\pi^9/f_\pi$, and so $\mathcal{L} = -\frac14[({\rm Tr}\,1)f_\pi^2 + 9F^2]f_9^{-2}\partial^\mu\pi^9\partial_\mu\pi^9$. Requiring the coefficient of $\partial^\mu\pi^9\partial_\mu\pi^9$ to be $-\frac12$ yields $F^2 = \frac19(2f_9^2 - 3f_\pi^2)$.

b) Only the mass terms for π^0, η, and π^9 are different. They are now

$$\begin{aligned}
\mathcal{L}_{\rm mass} = &-2v^3 f_\pi^{-2}[m_u(\tfrac{1}{\sqrt3}\eta + \pi^0 + r\pi^9)^2 + m_d(\tfrac{1}{\sqrt3}\eta - \pi^0 + r\pi^9)^2 \\
&+ m_s(\tfrac{2}{\sqrt3}\eta + r\pi^9)^2]\ , \qquad (83.53)
\end{aligned}$$

where $r \equiv f_\pi/f_9$. Setting $m_u = m_d \equiv m$, we get

$$\mathcal{L}_{\rm mass} = -2v^3 f_\pi^{-2}[2m(\pi^0)^2 + 2m(\tfrac{1}{\sqrt3}\eta + r\pi^9)^2 + m_s(\tfrac{2}{\sqrt3}\eta + r\pi^9)^2]\ . \qquad (83.54)$$

So we have $m_\pi^2 = 4mv^3/f_\pi^2$ as before. In the limit $m \ll m_s$, the eigenvalues of the η-π^9 mass-squared matrix are $m_\eta^2 = \frac83 m_s(1 + \frac34 r^2)v^3 f_\pi^{-2}$ and

$$m_9^2 = \frac{9r^2}{4 + 3r^2}\,m_\pi^2\ . \qquad (83.55)$$

Thus the maximum possible value of m_9 is $\sqrt3\,m_\pi$, attained in the limit $f_9 \to 0$.

83.8) We have

$$\begin{aligned}
\mathcal{L} = &-c_1\overline{N}(MP_{\rm L} + M^\dagger P_{\rm R})N - c_2\overline{N}(U^\dagger M^\dagger U^\dagger P_{\rm L} + UMUP_{\rm R})N \\
&- c_3\,{\rm Tr}(MU + M^\dagger U^\dagger)\,\overline{N}(U^\dagger P_{\rm L} + UP_{\rm R})N \\
&- c_4\,{\rm Tr}(MU - M^\dagger U^\dagger)\,\overline{N}(U^\dagger P_{\rm L} - UP_{\rm R})N \\
= &-c_1\overline{\mathcal{N}}(uMuP_{\rm L} + u^\dagger M^\dagger u^\dagger P_{\rm R})\mathcal{N} - c_2\overline{\mathcal{N}}(u^\dagger M^\dagger u^\dagger P_{\rm L} + uMuP_{\rm R})\mathcal{N} \\
&- c_3\,{\rm Tr}(MU + M^\dagger U^\dagger)\,\overline{\mathcal{N}}\mathcal{N} - c_4\,{\rm Tr}(MU - M^\dagger U^\dagger)\,\overline{\mathcal{N}}\gamma_5\mathcal{N} \\
= &-\tfrac12 c_+\overline{\mathcal{N}}(uMu + u^\dagger M^\dagger u^\dagger)\mathcal{N} + \tfrac12 c_-\overline{\mathcal{N}}(uMu - u^\dagger M^\dagger u^\dagger)\gamma_5\mathcal{N} \\
&- c_3\,{\rm Tr}(MU + M^\dagger U^\dagger)\,\overline{\mathcal{N}}\mathcal{N} - c_4\,{\rm Tr}(MU - M^\dagger U^\dagger)\,\overline{\mathcal{N}}\gamma_5\mathcal{N}\ , \qquad (83.56)
\end{aligned}$$

where $c_\pm = c_1 \pm c_2$, and $c_{1,2,3}$ are numerical coefficients. Now set $u = 1$; then the first term contributes $c_+ M$ to the nucleon mass matrix, and hence makes a contribution of $c_+(m_u - m_d)$ to the proton-neutron mass difference, $m_p - m_n = -1.3\,$MeV. Using $m_u = 1.7\,$MeV and

$m_d = 3.9\,\text{MeV}$ yields $c_+ = 0.6$. However, the electomagnetic mass of the proton is comparable in size to the proton-neutron mass difference, so a better estimate of c_+ comes from the masses of baryons with strange quarks, $c_+(m_s - \frac{1}{2}m_u - \frac{1}{2}m_d) = m_{\Xi^0} - m_{\Sigma^0} = 122\,\text{MeV}$; using $m_s = 76\,\text{MeV}$ yields $c_+ = 1.7$. We will need this number in section 94.

84 Spontaneous Breaking of Gauge Symmetries

84.1) a) We have $V = \frac{1}{2}m^2v^2\sum_i \alpha_i^2 + \frac{1}{4}\lambda_1 v^4 \sum_i \alpha_i^4 + \frac{1}{4}v^4\lambda_2(\sum_i \alpha_i^2)^2$. Differentiating with respect to v and imposing $\sum_i \alpha_i^2 = 1$ yields $m^2v + [\lambda_1 A(\alpha) + \lambda_2 B(\alpha)]v^3$, where $A(\alpha) \equiv \sum_i \alpha_i^4$ and $B(\alpha) \equiv 1$. Setting this to zero, solving for v, and plugging back into V yields eq. (84.18).

b) The coefficient of v^4 in V is $\lambda_1 A(\alpha)+\lambda_2 B(\alpha)$, so if this is negative the v^4 term is negative, and becomes arbitrarily large for large v.

c) Since $\lambda_1 A(\alpha)+\lambda_2 B(\alpha)$ must be positive, and since V is proportional to its negative inverse, making $\lambda_1 A(\alpha) + \lambda_2 B(\alpha)$ as small as possible will make V as negative as possible.

d) We want to extremize V, and hence $\lambda_1 A(\alpha) + \lambda_2 B(\alpha)$, and hence $\sum_i \alpha_i^4$. To impose the constraints $\sum_i \alpha_i^2 = 1$ and $\sum_i \alpha_i = 0$, we extremize $\sum_i(\frac{1}{4}\alpha_i^4 + \frac{1}{2}a\alpha_i^2 + b\alpha_i)$, where a and b are Lagrange multipliers. This yields a cubic equation for each α_i, $\alpha_i^3 + a\alpha_i + b = 0$, which has at most three different solutions. The sum of these roots equals minus the coefficient of the quadratic term, which is zero.

e) Recall that any set of N numbers x_i with mean $\bar{x} = N^{-1}\sum_i x_i$ obeys $\sum_i(x_i - \bar{x})^2 \geq 0$ or equivalently $\sum_i x_i^2 \geq N\bar{x}^2 = N^{-1}(\sum_i x_i)^2$. Letting $x_i = \alpha_i^2$ we have $\sum_i \alpha_i^4 \geq N^{-1}(\sum_i \alpha_i^2)^2 = N^{-1}$. This inequality is saturated by (and only by) $\alpha_i = \pm N^{-1/2}$. To have $\sum_i \alpha_i = 0$ is then possible only if N is even, and only if there are equal numbers of plus and minus signs; that is, $N_+ = N_- = \frac{1}{2}N$.

For N odd, the inequality cannot be saturated, and so things are more complicated; see L. F. Li, Phys. Rev. D 9, 1723 (1974), Appendix B. The following simplified analysis is due to Richard Eager.

First assume that only two of the three allowed values of the α_i's occur; call these two values β_+ and β_-. We suppose that $\beta_\pm$ occurs $N_\pm$ times, with $N_+ + N_- = N$. We then have $\sum_i \alpha_i = N_+\beta_+ + N_-\beta_- = 0$ and $\sum_i \alpha_i^2 = N_+\beta_+^2 + N_-\beta_-^2 = 1$, which implies $\beta_\pm^2 = N_\mp/N_\pm N$. Letting $N_+ = \frac{1}{2}(N+\Delta)$ and $N_- = \frac{1}{2}(N-\Delta)$, we find $\sum_i \alpha_i^4 = N_+\beta_+^4 + N_-\beta_-^4 = (N^2+3\Delta^2)/(N^3-N\Delta^2)$, which is a monotonically increasing function of Δ; therefore the minimum is achieved for the smallest possible value of Δ, which is zero for even N and one for odd N. For odd N the minimum value of $\sum_i \alpha_i^4$ is then $(N^2+3)/(N^3-N)$.

Now suppose that all three possible values of the α_i's appear; call these values β_+, β_-, and β_0. We will show that $\sum_i \alpha_i^4$ is larger than $(N^2+3)/(N^3-N)$, its minimum value when only β_+ and β_- appear. Hence the solution with only β_+ and β_- is preferred.

Label the α_i's so that $\alpha_1 = \beta_+$, $\alpha_2 = \beta_-$, and $\alpha_3 = \beta_0$. Let $r \equiv \beta_+^2 + \beta_-^2 + \beta_0^2$. Then we have $\sum_{i=4}^N \alpha_i^2 = 1-r$, and so $\sum_{i=4}^N \alpha_i^4 \geq (N-3)^{-1}(\sum_{i=4}^N \alpha_i^2)^2 = (N-3)^{-1}(1-r)^2$. From part (d), we have $\beta_+ + \beta_- + \beta_0 = 0$. An identity satisfied by any three numbers that sum to zero is $\beta_+^4 + \beta_-^4 + \beta_0^4 = \frac{1}{2}(\beta_+^2 + \beta_-^2 + \beta_0^2)^2$, which in our case becomes $\beta_+^4 + \beta_-^4 + \beta_0^4 = \frac{1}{2}r^2$. Therefore $\sum_{i=1}^N \alpha_i^4 = \frac{1}{2}r^2 + \sum_{i=4}^N \alpha_i^4 \geq \frac{1}{2}r^2 + (N-3)^{-1}(1-r)^2$. Minimizing the right-hand side with respect to r, we get $r = 2/(N-1)$ and hence $\sum_{i=1}^N \alpha_i^4 \geq 1/(N-1)$, which is larger than $(N^2+3)/(N^3-N)$ for $N > 3$.

For $N = 3$, minima with all three values appearing have the same energy as minima with only two. This is an accidental degeneracy that is lifted by quantum corrections.

85 Spontaneously Broken Abelian Gauge Theory

86 Spontaneously Broken Nonabelian Gauge Theory

86.1) a) Let $\phi'_i \equiv \phi_{i+d(\mathrm{R})}$, so that $\varphi_i = \frac{1}{\sqrt{2}}(\phi_i + i\phi'_i)$. Also, let $R^a \equiv \mathrm{Re}\, T^a_{\mathrm{R}}$ and $J^a \equiv \mathrm{Im}\, T^a_{\mathrm{R}}$. Substituting these into $\delta\varphi_i = -i\theta^a (T^a_{\mathrm{R}})_i{}^j \varphi_j$, we get

$$\begin{aligned} \delta\phi_i + i\delta\phi'_i &= -i\theta^a[(R^a)_i{}^j + i(J^a)_i{}^j](\phi_j + i\phi'_j) \\ &= \theta^a[(J^a)_i{}^j\phi_j + (R^a)_i{}^j\phi'_j] + i\theta^a[-(R^a)_i{}^j\phi_j + (J^a)_i{}^j\phi'_j] \,. \end{aligned} \tag{86.28}$$

This can be written as

$$\begin{pmatrix} \delta\phi \\ \delta\phi' \end{pmatrix} = -i\theta^a \begin{pmatrix} iJ^a & iR^a \\ -iR^a & iJ^a \end{pmatrix} \begin{pmatrix} \phi \\ \phi' \end{pmatrix}. \tag{86.29}$$

Thus we identify

$$\mathcal{T}^a = i \begin{pmatrix} J^a & R^a \\ -R^a & J^a \end{pmatrix}. \tag{86.30}$$

b) From eq. (86.30), we have

$$\mathcal{T}^a\mathcal{T}^b = -\begin{pmatrix} J^aJ^b - R^aR^b & J^aR^b + R^aJ^b \\ -R^aJ^b - J^aR^b & -R^aR^b + J^aJ^b \end{pmatrix}, \tag{86.31}$$

and hence

$$[\mathcal{T}^a, \mathcal{T}^b] = \begin{pmatrix} [R^a, R^b] - [J^a, J^b] & -[R^a, J^b] - [J^a, R^b] \\ [R^a, J^b] + [J^a, R^b] & [R^a, R^b] - [J^a, J^b] \end{pmatrix}. \tag{86.32}$$

From $[T^a_{\mathrm{R}}, T^b_{\mathrm{R}}] = if^{abc}T^c_{\mathrm{R}}$, we have $[R^a + iJ^a, R^b + iJ^b] = if^{abc}(R^c + iJ^c)$. Collecting the real and imaginary parts on each side, we find $[R^a, R^b] - [J^a, J^b] = -f^{abc}J^c$ and $[R^a, J^b] + [J^a, R^b] = f^{abc}R^c$. Using these in eq. (86.32), we find

$$[\mathcal{T}^a, \mathcal{T}^b] = if^{abc} \begin{pmatrix} iJ^c & iR^c \\ -iR^c & iJ^c \end{pmatrix}, \tag{86.33}$$

and hence $[\mathcal{T}^a, \mathcal{T}^b] = if^{abc}\mathcal{T}^c$.

87 The Standard Model: Gauge and Higgs Sector

87.1) See eq. (88.15) and eq. (88.16).

87.2) a) $e = (4\pi/127.9)^{1/2} = 0.313$, $g_2 = e/s_{\rm W} = 0.652$, $g_1 = e/c_{\rm W} = 0.357$, $v = 2M_{\rm W}/g_2 = 247\,{\rm GeV}$.

b) $G_{\rm F} = \pi\alpha/\sqrt{2}\sin^2\theta_{\rm W} M_{\rm W}^2 = 1.16\times 10^{-5}\,{\rm GeV}^2$. Actual value from muon decay is $1.166\times 10^{-5}\,{\rm GeV}^2$.

c) $M_{\rm W} = \frac{1}{2}g_2 v = ev/2s_{\rm W}$, so $G_{\rm F} = 1/\sqrt{2}v^2$. Thus a measurement of $G_{\rm F}$ is a direct measurement of the Higgs vacuum expectation value.

87.3) a) From problem 86.1 we have

$$\mathcal{T}^a = i\begin{pmatrix} J^a & R^a \\ -R^a & J^a \end{pmatrix}, \tag{87.28}$$

where $R^a \equiv {\rm Re}\,T^a$ and $J^a \equiv {\rm Im}\,T^a$. This yields

$$\mathcal{T}^1 = \frac{i}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{T}^2 = \frac{i}{2}\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\mathcal{T}^3 = \frac{i}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad \mathcal{Y} = \frac{i}{2}\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{87.29}$$

b) We have $F^a{}_i = ig_a(\mathcal{T}^a)_{ij}v_j$, where $v_j = v\delta_{j1}$ in our case. Letting $\mathcal{T}^4 \equiv \mathcal{Y}$, we have $g_a \to g_2$ for $a = 1,2,3$ and $g_a \to g_1$ for $a = 4$. We find

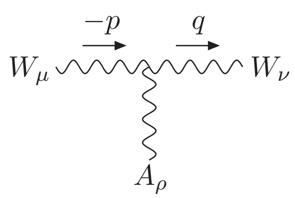
$$F^a{}_i = \frac{v}{2}\begin{pmatrix} 0 & 0 & 0 & g_2 \\ 0 & -g_2 & 0 & 0 \\ 0 & 0 & g_2 & 0 \\ 0 & 0 & -g_1 & 0 \end{pmatrix}. \tag{87.30}$$

c) We have $F^a{}_i F^b{}_i = \frac{1}{4}v^2\Big(\cdots\Big)\Big(\cdots\Big)^{\rm T}$, where $\Big(\cdots\Big)$ denotes the matrix in eq. (87.30). We get

$$F^a{}_i F^b{}_i = \frac{v^2}{4}\begin{pmatrix} g_2^2 & 0 & 0 & 0 \\ 0 & g_2^2 & 0 & 0 \\ 0 & 0 & g_2^2 & -g_1 g_2 \\ 0 & 0 & -g_1 g_2 & g_1^2 \end{pmatrix}. \tag{87.31}$$

The eigenvalues are $\frac{1}{4}g_2^2v^2$, $\frac{1}{4}g_2^2v^2$, $\frac{1}{4}(g_1^2+g_2^2)v^2$, and 0, corresponding to the W^+, W^-, Z^0, and photon.

87.4) Let's begin with the $WW\gamma$ vertex. Consider the third term on the first line of eq (87.27), $-(D^\mu W^{+\nu})^\dagger D_\mu W^+_\nu$. It has the structure of a kinetic term for a complex scalar field that carries an index ν. In analogy with fig. 61.1, the vertex factor for



would be $ie(-p+q)^\rho g^{\mu\nu}$. (The arrow direction corresponds to charge flow.) However, this vertex arises from the three-gauge-boson vertex in the SU(2) part of the gauge group, which has the structure of eq. (72.5). Thus the remaining terms in $\mathcal{L}$ that contribute to the $WW\gamma$ vertex must conspire to reproduce this structure. Thus we have a complete $WW\gamma$ vertex factor of

$$i\mathbf{V}^{\mu\nu\rho}_{WW\gamma}(p,q,r) = -ie[(p{-}q)^\rho g^{\mu\nu} + (q{-}r)^\mu g^{\nu\rho} + (r{-}p)^\nu g^{\rho\mu}] \,, \tag{87.32}$$

where $r = -p{-}q$ is the outgoing momentum of the photon.

Since $D_\mu = \partial_\mu - ie(A_\mu + \cot\theta_{\rm W} Z_\mu)$, the WWZ vertex is given by eq. (87.32) with $e \to e\cot\theta_{\rm W}$,

$$i\mathbf{V}^{\mu\nu\rho}_{WWZ}(p,q,r) = -i(e\cot\theta_{\rm W})[(p{-}q)^\rho g^{\mu\nu} + (q{-}r)^\mu g^{\nu\rho} + (r{-}p)^\nu g^{\rho\mu}] \,, \tag{87.33}$$

From the last two terms on the first line of eq (87.27), we see that the $\gamma\gamma WW$ interaction is

$$\begin{aligned}\mathcal{L}_{\gamma\gamma WW} &= -e^2(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})A_\mu A_\nu W^-_\rho W^+_\sigma \\ &= -\tfrac{1}{2}e^2(2g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho})A_\mu A_\nu W^-_\rho W^+_\sigma \,. \end{aligned} \tag{87.34}$$

Similarly, the γZWW and $ZZWW$ interactions are

$$\mathcal{L}_{\gamma ZWW} = -(e^2\cot\theta_{\rm W})(2g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho})A_\mu Z_\nu W^-_\rho W^+_\sigma \,, \tag{87.35}$$

$$\mathcal{L}_{ZZWW} = -\tfrac{1}{2}(e^2\cot^2\theta_{\rm W})(2g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho})Z_\mu Z_\nu W^-_\rho W^+_\sigma \,, \tag{87.36}$$

and from the third line of eq. (87.27),

$$\mathcal{L}_{WWWW} = +\tfrac{1}{4}(e^2/\sin^2\theta_{\rm W})(2g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho})W^+_\mu W^+_\nu W^-_\rho W^-_\sigma \,. \tag{87.37}$$

These yield the vertex factors

$$i\mathbf{V}^{\mu\nu\rho\sigma}_{\gamma\gamma WW} = -ie^2(2g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) \,, \tag{87.38}$$

$$i\mathbf{V}^{\mu\nu\rho\sigma}_{\gamma ZWW} = -i(e^2\cot\theta_{\rm W})(2g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) \,, \tag{87.39}$$

$$i\mathbf{V}^{\mu\nu\rho\sigma}_{ZZWW} = -i(e^2\cot^2\theta_{\rm W})(2g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) \,, \tag{87.40}$$

$$i\mathbf{V}^{\mu\nu\rho\sigma}_{WWWW} = +i(e^2/\sin^2\theta_{\rm W})(2g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) \,. \tag{87.41}$$

For the $WWWW$ vertex, the μ and ν lines have incoming charge arrows and the ρ and σ lines have outgoing charge arrows.

From the fourth line of eq. (87.27), we read off the vertex factors for interactions between the physical Higgs boson H and the gauge bosons,

$$i\mathbf{V}^{\mu\nu}_{HWW} = -2i(M_{\rm W}^2/v)g^{\mu\nu} \; , \tag{87.42}$$

$$i\mathbf{V}^{\mu\nu}_{HZZ} = -2i(M_{\rm Z}^2/v)g^{\mu\nu} \; , \tag{87.43}$$

$$i\mathbf{V}^{\mu\nu}_{HHWW} = -2i(M_{\rm W}^2/v^2)g^{\mu\nu} \; , \tag{87.44}$$

$$i\mathbf{V}^{\mu\nu}_{HHZZ} = -2i(M_{\rm Z}^2/v^2)g^{\mu\nu} \; . \tag{87.45}$$

From the last line of eq. (87.27), we read off the vertex factors for the self-interactions of H,

$$i\mathbf{V}_{3H} = -3i(m_{\rm H}^2/v) \; , \tag{87.46}$$

$$i\mathbf{V}_{4H} = -3i(m_{\rm H}^2/v^2) \; . \tag{87.47}$$

Since we did not include the unphysical Goldston boson explicitly, we are implicitly working in unitary gauge (equivalently, R_∞ gauge), and so the W and Z propagators are given by eq. (85.39),

$$\Delta^{\mu\nu}(k) = \frac{g^{\mu\nu} + k^\mu k^\nu/M^2}{k^2+M^2-i\epsilon} \; , \tag{87.48}$$

where M is $M_{\rm W}$ or $M_{\rm Z}$. The propagator for the physical Higgs boson is

$$\Delta(k^2) = \frac{1}{k^2+m_{\rm H}^2-i\epsilon} \; . \tag{87.49}$$

87.5) For $H \to W^+W^-$, the vertex factor is $-2i(M_{\rm W}^2/v)g^{\mu\nu}$, and thus the decay amplitude is $\mathcal{T} = -2(M_{\rm W}^2/v)(\varepsilon_1\cdot\varepsilon_2)$, where ε_1^μ and ε_2^μ are the outgoing W polarizations. (We drop primes on outgoing quantities for notational convenience.) Summing $|\mathcal{T}|^2$ over outgoing polarizations and using eq. (85.16), we get

$$\langle|\mathcal{T}|^2\rangle = \frac{4M_{\rm W}^4}{v^2}\left(g^{\mu\nu} + \frac{k_1^\mu k_1^\nu}{M_{\rm W}^2}\right)\left(g_{\mu\nu} + \frac{k_{2\mu}k_{2\nu}}{M_{\rm W}^2}\right) \; , \tag{87.50}$$

where $k_1^2 = k_2^2 = -M_{\rm W}^2$ and $2k_1\cdot k_2 = (k_1+k_2)^2 - k_1^2 - k_2^2 = -m_{\rm H}^2 + 2M_{\rm W}^2$. Thus we have

$$\begin{aligned}\langle|\mathcal{T}|^2\rangle &= \frac{4M_{\rm W}^4}{v^2}\left(4 - 1 - 1 + \frac{(k_1\cdot k_2)^2}{M_{\rm W}^4}\right) \\ &= \frac{m_{\rm H}^4}{v^2}\left(1 - \frac{4M_{\rm W}^2}{m_{\rm H}^2} + \frac{12M_{\rm W}^4}{m_{\rm H}^4}\right) .\end{aligned} \tag{87.51}$$

We then have

$$\begin{aligned}\Gamma_{H\to W^+W^-} &= \frac{\langle|\mathcal{T}|^2\rangle}{16\pi m_{\rm H}}\left(1 - \frac{4M_{\rm W}^2}{m_{\rm H}^2}\right) \\ &= \frac{m_{\rm H}^3}{16\pi v^2}\left(1 - \frac{4M_{\rm W}^2}{m_{\rm H}^2} + \frac{12M_{\rm W}^4}{m_{\rm H}^4}\right)\left(1 - \frac{4M_{\rm W}^2}{m_{\rm H}^2}\right) .\end{aligned} \tag{87.52}$$

Using $v = 246\,\text{GeV}$ and $M_{\text{W}} = 80.4\,\text{GeV}$, we get $\Gamma = 0.620\,\text{GeV}$ for $m_{\text{H}} = 200\,\text{GeV}$.

The calculation for $H \to Z^0 Z^0$ is identical, except that there is a symmetry factor of $S = 2$. So we get

$$\Gamma_{H\to Z^0 Z^0} = \frac{m_{\text{H}}^3}{32\pi v^2}\left(1 - \frac{4M_{\text{Z}}^2}{m_{\text{H}}^2} + \frac{12M_{\text{Z}}^4}{m_{\text{H}}^4}\right)\left(1 - \frac{4M_{\text{Z}}^2}{m_{\text{H}}^2}\right). \tag{87.53}$$

Using $v = 246\,\text{GeV}$ and $M_{\text{Z}} = 91.2\,\text{GeV}$, we get $\Gamma = 0.152\,\text{GeV}$ for $m_{\text{H}} = 200\,\text{GeV}$.

88 The Standard Model: Lepton Sector

88.1) We already checked all possible fermion mass terms in eq. (88.4). To get an allowed Yukawa coupling, we note that the only scalar field is a 2 of SU(2), and that $2\otimes 2\otimes 2$ and $2\otimes 1\otimes 1$ do not contain a 1, so the only possible Yukawa couplings allowed by SU(2) are $\varphi\ell\bar e$ and $\varphi^\dagger\ell\bar e$. We have $\varphi^\dagger\sim(2,+\frac12)$, and so the sum of the hypercharges is not zero for the second possibility; thus it is not allowed. Finally, adding more fields increases the dimension beyond four, so there are no other terms to consider. Q.E.D.

88.2) This follows immediately from eq. (75.8) for $P_{\rm L}\Psi$. Neutrinos are created by $b^\dagger$ operators, and antineutrinos by $d^\dagger$ operators. Eq. (75.8) shows that a particle created by a $b^\dagger$ must have helicity $-\frac12$, while a particle created by a $d^\dagger$ must have helicity $+\frac12$.

88.3) Written in terms of fields with definite mass, eq. (88.33) contains a factor of $\sum_I y_I\varphi\ell_I\bar e_I$, where gauge-group indices have been suppressed. This is invariant under a global transformation $\ell_I\to e^{-i\alpha_I}\ell_I$, $\bar e_I\to e^{+i\alpha_I}\bar e_I$, with an independent phase α_I for each generation. Eq. (88.32) is also invariant under this transformation. The Dirac fields $\mathcal{E}_I$ and $\mathcal{N}_{{\rm L}I}$ each have charge +1 under the transformation associated with that generation, and charge zero under the other two transformations. So the electron and electron-neutrino have electron number +1, and muon and tau number zero.

88.4) The amplitude that follows from eq. (88.36) is $\mathcal{T}=2\sqrt2 G_{\rm F}(\overline u{}'_3\gamma^\mu P_{\rm L}v'_2)(\overline u{}'_1\gamma_\mu P_{\rm L}u_1)$, and, using $\overline{\gamma_\mu P_{\rm L}}=\gamma_\mu P_{\rm L}$, its complex conjugate is $\mathcal{T}^*=2\sqrt2 G_{\rm F}(\overline v{}'_2\gamma^\nu P_{\rm L}u'_3)(\overline u_1\gamma_\nu P_{\rm L}u'_1)$. Summing over the final spins and averaging over the initial spin, we have

$$\begin{aligned}\langle|\mathcal{T}|^2\rangle &= \tfrac12(2\sqrt2)^2G_{\rm F}^2\,{\rm Tr}[(-\not p_1+m_\mu)\gamma_\nu P_{\rm L}(-\not p'_1)\gamma_\mu P_{\rm L}]\,{\rm Tr}[(-\not p'_2)\gamma^\nu P_{\rm L}(-\not p'_3+m_e)\gamma^\mu P_{\rm L}]\\ &= G_{\rm F}^2\,{\rm Tr}[\not p_1\gamma_\nu\not p'_1\gamma_\mu(1-\gamma_5)]\,{\rm Tr}[\not p'_2\gamma^\nu\not p'_3\gamma^\mu(1-\gamma_5)]\\ &= 16G_{\rm F}^2[p_{1\nu}p'_{1\mu}+p_{1\mu}p'_{1\nu}-(p_1p'_1)g_{\mu\nu}+i\varepsilon_{\alpha\nu\beta\mu}p_1^\alpha p_1'^\beta]\\ &\qquad\times[p_2'^\nu p_3'^\mu+p_2'^\mu p_3'^\nu-(p'_2p'_3)g^{\mu\nu}+i\varepsilon^{\gamma\nu\delta\mu}p'_{2\gamma}p'_{3\delta}]\\ &= 16G_{\rm F}^2([p_{1\nu}p'_{1\mu}+p_{1\mu}p'_{1\nu}-(p_1p'_1)g_{\mu\nu}][p_2'^\nu p_3'^\mu+p_2'^\mu p_3'^\nu-(p'_2p'_3)g^{\mu\nu}]\\ &\qquad-\varepsilon_{\alpha\nu\beta\mu}\varepsilon^{\gamma\nu\delta\mu}p_1^\alpha p_1'^\beta p'_{2\gamma}p'_{3\delta})\\ &= 16G_{\rm F}^2[2(p_1p'_2)(p'_1p'_3)+2(p_1p'_3)(p'_1p'_2)-2(p'_2p'_3)(p_1p'_1)-2(p_1p'_1)(p'_2p'_3)+4(p_1p'_1)(p'_2p'_3)\\ &\qquad+2(\delta_\alpha{}^\gamma\delta_\beta{}^\delta-\delta_\alpha{}^\delta\delta_\beta{}^\gamma)p_1^\alpha p_1'{}^\beta p'_{2\gamma}p'_{3\delta}]\\ &= 32G_{\rm F}^2[(p_1p'_2)(p'_1p'_3)+(p_1p'_3)(p'_1p'_2)\\ &\qquad+(p_1p'_2)(p'_1p'_3)-(p_1p'_3)(p'_1p'_2)]\\ &= 64G_{\rm F}^2(p_1p'_2)(p'_1p'_3)\;. \end{aligned}\tag{88.50}$$

88.5) a) Only the neutral current contributes to $\nu_\mu e^-\to\nu_\mu e^-$, so we have $\mathcal{L}_{\rm eff}=2\sqrt2\,G_{\rm F}J_{\rm Z}^\mu J_{{\rm Z}\mu}$, with the relevant terms in the current given by

$$\begin{aligned}J_{\rm Z}^\mu &= J_3^\mu-s_{\rm W}^2J_{\rm EM}^\mu\\ &= \tfrac14\overline{\mathcal{N}}\gamma^\mu(1-\gamma_5)\mathcal{N}-\tfrac14\overline{\mathcal{E}}\gamma^\mu(1-\gamma_5)\mathcal{E}+s_{\rm W}^2\overline{\mathcal{E}}\gamma^\mu\mathcal{E}\\ &= \tfrac14\overline{\mathcal{N}}\gamma^\mu(1-\gamma_5)\mathcal{N}+\tfrac12\overline{\mathcal{E}}\gamma^\mu[(-\tfrac12+2s_{\rm W}^2)+\tfrac12\gamma_5]\mathcal{E}\;.\end{aligned}\tag{88.51}$$

Thus we find $\mathcal{L}_{\rm eff} = \frac{1}{\sqrt{2}}G_{\rm F}\overline{\mathcal{N}}\gamma^\mu(1-\gamma_5)\mathcal{N}\,\overline{\mathcal{E}}\gamma_\mu(C_{\rm V}-C_{\rm A}\gamma_5)\mathcal{E}$ with $C_{\rm V} = -\frac{1}{2}+2s^2_{\rm W}$ and $C_{\rm A} = -\frac{1}{2}$.

b) Both the charged and neutral currents contribute to $\nu_e e^- \to \nu_e e^-$. The neutral current analysis is the same as above. The extra contribution to $\mathcal{L}_{\rm eff}$ from the charged current is

$$\begin{aligned}\Delta\mathcal{L}_{\rm eff} &= 2\sqrt{2}\,G_{\rm F}J^{+\mu}J^-_\mu \\ &= \tfrac{1}{\sqrt{2}}G_{\rm F}\overline{\mathcal{E}}\gamma^\mu(1-\gamma_5)\mathcal{N}\,\overline{\mathcal{N}}\gamma_\mu(1-\gamma_5)\mathcal{E} \\ &= \tfrac{1}{\sqrt{2}}G_{\rm F}\overline{\mathcal{N}}\gamma^\mu(1-\gamma_5)\mathcal{N}\,\overline{\mathcal{E}}\gamma_\mu(1-\gamma_5)\mathcal{E}\ , \end{aligned} \tag{88.52}$$

where the last line follows from the Fierz identity, eq (36.62). We see that $\Delta C_{\rm V} = \Delta C_{\rm A} = 1$, and so $C_{\rm V} = \frac{1}{2} + s^2_{\rm W}$ and $C_{\rm A} = \frac{1}{2}$.

c) $\mathcal{T} = \frac{1}{\sqrt{2}}G_{\rm F}\overline{u}'_\nu\gamma^\alpha(1-\gamma_5)u_\nu\overline{u}'_e\gamma_\alpha(C_{\rm V}-C_{\rm A}\gamma_5)u_e$, $\mathcal{T}^* = \frac{1}{\sqrt{2}}G_{\rm F}\overline{u}_\nu\gamma^\beta(1-\gamma_5)u'_\nu\overline{u}_e\gamma_\beta(C_{\rm V}-C_{\rm A}\gamma_5)u'_e$,

$$\begin{aligned}\langle|\mathcal{T}|^2\rangle &= \tfrac{1}{4}(\tfrac{1}{\sqrt{2}}G_{\rm F})^2\,{\rm Tr}[(-\not p_\nu)\gamma^\beta(1-\gamma_5)(-\not p'_\nu)\gamma^\alpha(1-\gamma_5)] \\ &\qquad\times{\rm Tr}[(-\not p_e+m_e)\gamma_\beta(C_{\rm V}-C_{\rm A}\gamma_5)(-\not p'_e+m_e)\gamma_\alpha(C_{\rm V}-C_{\rm A}\gamma_5)]\ . \end{aligned} \tag{88.53}$$

Let's evaluate the second trace:

$$\begin{aligned}{\rm Tr}[\ldots] &= {\rm Tr}[\not p_e\gamma_\beta(C_{\rm V}-C_{\rm A}\gamma_5)\not p'_e\gamma_\alpha(C_{\rm V}-C_{\rm A}\gamma_5)] + m_e^2\,{\rm Tr}[\gamma_\beta(C_{\rm V}-C_{\rm A}\gamma_5)\gamma_\alpha(C_{\rm V}-C_{\rm A}\gamma_5)] \\ &= {\rm Tr}[\not p_e\gamma_\beta\not p'_e\gamma_\alpha(C_{\rm V}-C_{\rm A}\gamma_5)^2] + m_e^2\,{\rm Tr}[\gamma_\beta\gamma_\alpha(C_{\rm V}+C_{\rm A}\gamma_5)(C_{\rm V}-C_{\rm A}\gamma_5)] \\ &= {\rm Tr}[\not p_e\gamma_\beta\not p'_e\gamma_\alpha(C^2_{\rm V}+C^2_{\rm A}-2C_{\rm V}C_{\rm A}\gamma_5)] + m_e^2\,{\rm Tr}[\gamma_\beta\gamma_\alpha(C^2_{\rm V}+C^2_{\rm A})] \\ &= 4(C^2_{\rm V}+C^2_{\rm A})[p_{e\beta}p'_{e\alpha} + p_{e\alpha}p'_{e\beta} - (p_ep'_e)g_{\alpha\beta}] + 8iC_{\rm V}C_{\rm A}\varepsilon_{\rho\beta\sigma\alpha}p_e^\rho p_e'^\sigma \\ &\quad - 4(C^2_{\rm V}+C^2_{\rm A})m_e^2 g_{\alpha\beta} \\ &= 4(C^2_{\rm V}+C^2_{\rm A})[p_{e\beta}p'_{e\alpha} + p_{e\alpha}p'_{e\beta} - (p_ep'_e+m_e^2)g_{\alpha\beta}] + 8iC_{\rm V}C_{\rm A}\varepsilon_{\rho\beta\sigma\alpha}p_e^\rho p_e'^\sigma\ . \end{aligned} \tag{88.54}$$

We get the first trace from this one via $e \to \nu$, $m_e \to 0$, $C_{\rm V} \to 1$, and $C_{\rm A} \to 1$, so the first trace is

$${\rm Tr}[\ldots] = 8[p_\nu^\beta p_\nu'^\alpha + p_\nu^\alpha p_\nu'^\beta - (p_\nu p'_\nu)g^{\alpha\beta}] + 8i\varepsilon^{\lambda\beta\kappa\alpha}p_{\nu\lambda}p'_{\nu\kappa}\ . \tag{88.55}$$

Now we have

$$\begin{aligned}\langle|\mathcal{T}|^2\rangle &= 8G_{\rm F}^2\Big[(C^2_{\rm V}+C^2_{\rm A})\Big((p_ep_\nu)(p'_ep'_\nu) + (p_ep'_\nu)(p'_ep_\nu) + m_e^2(p_\nu p'_\nu)\Big) \\ &\qquad + 2C_{\rm V}C_{\rm A}\Big((p_ep_\nu)(p'_ep'_\nu) - (p_ep'_\nu)(p'_ep_\nu)\Big)\Big]\ . \end{aligned} \tag{88.56}$$

Using $p_ep_\nu = p'_ep'_\nu = -\frac{1}{2}(s-m_e^2)$, $p_\nu p'_\nu = \frac{1}{2}t = -\frac{1}{2}(s+u-2m_e^2)$, $p_ep'_\nu = p'_ep_\nu = \frac{1}{2}(u-m_e^2)$, we get

$$\langle|\mathcal{T}|^2\rangle = 2G_{\rm F}^2\Big[(C^2_{\rm V}+C^2_{\rm A})\Big(s^2+u^2-4m_e^2(s+u)+6m_e^4\Big)+2C_{\rm V}C_{\rm A}\Big(s^2-u^2-2m_e^2(s-u)\Big)\Big]. \tag{88.57}$$

88.6) Consider a massive vector field Z^μ and a Dirac fermion field Ψ with $\mathcal{L}_{\rm int} = Z^\mu\overline{\Psi}(g_{\rm V}-g_{\rm A}\gamma_5)\Psi$; then the amplitude for $Z \to e^+e^-$ is $\mathcal{T} = \varepsilon^{*\mu}\overline{v}_2\gamma_\mu(g_{\rm V}-g_{\rm A}\gamma_5)u_1$. (We have dropped the primes on outgoing quantities for notational convenience). The amplitude is the same if $\overline{\Psi}$ is a different Dirac field that is unrelated to Ψ, so it also holds for a process like $W^+ \to e^+\overline{\nu}$. So, first we will compute the decay rate without specifying $g_{\rm V}$ and $g_{\rm A}$, and then we will find the values of $g_{\rm V}$ and $g_{\rm A}$ for the three processes of interest, $Z^0 \to e^+e^-$, $Z^0 \to \overline{\nu}_e\nu_e$, and $W^+ \to e^+\overline{\nu}$. We will neglect the electron mass.

We have $\mathcal{T}^* = \varepsilon^\nu\,\overline{u}_1\gamma_\nu(g_{\rm V}-g_{\rm A}\gamma_5)v_2$, and so, summing over final spins and averaging over the three initial polarizations, we have

$$\begin{aligned}\langle|\mathcal{T}|^2\rangle &= \tfrac{1}{3}(\textstyle\sum_{\rm pol}\varepsilon^\nu\varepsilon^{*\mu})\,{\rm Tr}[\not{p}_1\gamma_\nu(g_{\rm V}-g_{\rm A}\gamma_5)\not{p}_2\gamma_\mu(g_{\rm V}-g_{\rm A}\gamma_5)] \\ &= \tfrac{1}{3}(g^{\mu\nu}+k^\mu k^\nu/M^2)\,{\rm Tr}[\not{p}_1\gamma_\nu(g_{\rm V}-g_{\rm A}\gamma_5)\not{p}_2\gamma_\mu(g_{\rm V}-g_{\rm A}\gamma_5)]\;, \end{aligned}\tag{88.58}$$

where $k = p_1 + p_2$ is the momentum of the vector particle, and M is its mass. We evaluated this trace in problem 88.5; we then have

$$\begin{aligned}\langle|\mathcal{T}|^2\rangle &= \tfrac{8}{3}(g_{\rm V}^2+g_{\rm A}^2)(g^{\mu\nu}+k^\mu k^\nu/M^2)[p_{1\mu}p_{2\nu}+p_{1\nu}p_{2\mu}-(p_1p_2)g_{\mu\nu}] \\ &= \tfrac{4}{3}(g_{\rm V}^2+g_{\rm A}^2)[(1+1-4+1)(p_1p_2)+2(kp_1)(kp_2)/M^2]\;. \end{aligned}\tag{88.59}$$

We have $k^2 = (p_1+p_2)^2 = -M^2$ and $p_1^2 = p_2^2 = 0$, so $p_1p_2 = -\frac{1}{2}M^2$. Also $kp_i = (p_1+p_2)p_i = p_1p_2 = -\frac{1}{2}M^2$. Thus the factor in square brackets evaluates to M^2, and so

$$\langle|\mathcal{T}|^2\rangle = \tfrac{4}{3}(g_{\rm V}^2+g_{\rm A}^2)M^2\;. \tag{88.60}$$

For distinguishable outgoing massless particles, we get the total decay rate by dividing by $16\pi M$; thus

$$\Gamma = \tfrac{1}{12\pi}(g_{\rm V}^2+g_{\rm A}^2)M\;. \tag{88.61}$$

Now we consider our three processes. From eqs. (88.23) and (88.25), we see that for $W^+ \to e^+\overline{\nu}_e$, we have $g_{\rm V} = g_{\rm A} = g_2/2\sqrt{2}$; using $g_2 = e/s_{\rm W}$ and $e^2 = 4\pi\alpha$, we get

$$\Gamma_{W^+\to e^+\overline{\nu}_e} = \frac{\alpha}{12s_{\rm W}^2}M_{\rm W}\;. \tag{88.62}$$

For $Z^0 \to \overline{\nu}_e\nu_e$, we have $g_{\rm V} = g_{\rm A} = e/4s_{\rm W}c_{\rm W}$, and so

$$\Gamma_{Z^0\to\overline{\nu}_e\nu_e} = \frac{\alpha}{24s_{\rm W}^2c_{\rm W}^2}M_{\rm Z}\;. \tag{88.63}$$

For $Z^0 \to e^+e^-$, we have $g_{\rm V} = (-\frac{1}{4}+s_{\rm W}^2)e/s_{\rm W}c_{\rm W}$ and $g_{\rm A} = -\frac{1}{4}e/s_{\rm W}c_{\rm W}$, so

$$\Gamma_{Z^0\to e^+e^-} = \frac{\alpha}{24s_{\rm W}^2c_{\rm W}^2}(1-4s_{\rm W}^2+8s_{\rm W}^4)M_{\rm Z}\;. \tag{88.64}$$

Putting in numbers ($\alpha = 1/127.9$, $s_{\rm W}^2 = 0.231$, $M_{\rm W} = 80.4\,{\rm GeV}$, $M_{\rm Z} = 91.2\,{\rm GeV}$), we find $\Gamma_{W^+\to e^+\overline{\nu}_e} = 0.227\,{\rm GeV}$, $\Gamma_{Z^0\to\overline{\nu}_e\nu_e} = 0.167\,{\rm GeV}$, and $\Gamma_{Z^0\to e^+e^-} = 0.084\,{\rm GeV}$. (These predictions are in excellent agreement with experiment.)

88.7) a) With all parameters given as $\overline{\rm MS}$ parameters at some particular scale, any derived quantity is also an $\overline{\rm MS}$ parameters at that scale. Here $M_{\rm W}$ should really be the $\overline{\rm MS}$ parameter $M_{\rm W}(\mu)$ with $\mu = M_{\rm Z}$, but the difference between this and the physical W mass is small and can be neglected.

b) We have $G_{\rm F0} = Z_G G_{\rm F}/(\prod_{i=1}^4 Z_i^{1/2})$, where Z_i is the renormalizing factor for the kinetic term of each of the four fermion fields. In the present case, since we are considering only electromagnetic effects, and since two fields have charge zero and two have charge one, $\prod_{i=1}^{4} Z_i^{1/2} = Z_2$, where Z_2 is the renormalizing factor for a Dirac field of charge one. Taking the logarithm, we get $\ln G_{\rm F0} = \ln G_{\rm F} + \mathcal{G}(\alpha,\varepsilon)$, where $\ln(Z_G/Z_2) \equiv \mathcal{G}(\alpha,\varepsilon) = \sum_{n=1}^{\infty} \mathcal{G}_n(\alpha)/\varepsilon^n$. Taking $d/d\ln\mu$, we get $0 = G_{\rm F}^{-1} dG_{\rm F}/d\ln\mu + (\partial G/\partial\alpha)\partial\alpha/d\ln\mu$. Using $\partial\alpha/d\ln\mu = -\varepsilon\alpha + \beta(\alpha)$, rearranging, and dropping negative powers of ε (because their coefficients must work out to be zero), we get $dG_{\rm F}/d\ln\mu = \alpha\mathcal{G}_1'(\alpha)G_{\rm F}$.

c) Let $t \equiv \ln\mu$. Then we have $dG_{\rm F}/G_{\rm F} = \gamma_G(\alpha)dt$ and $d\alpha/\beta(\alpha) = dt$, so $dG_{\rm F}/G_{\rm F} = (\gamma_G/\beta)d\alpha = (c_1/b_1)d\alpha/\alpha$. Integrating, we get $\ln[G_{\rm F}(\mu_1)/G_{\rm F}(\mu_2)] = (c_1/b_1)\ln[\alpha(\mu_1)/\alpha(\mu_2)]$, which yields eq. (88.47) after setting $\mu_1 = \mu$ and $\mu_2 = M_{\rm W}$.

d) For $\beta(\alpha) = b_1\alpha^2$, integrating $d\alpha/\beta(\alpha) = dt$ yields $\alpha(M_{\rm W}) = [1 + b_1\alpha(\mu)\ln(M_{\rm W}/\mu)]\alpha(\mu)$; plugging this into eq. (88.47) and expanding in $\alpha(\mu)\ln(M_{\rm W}/\mu)$ yields eq. (88.48).

e) This is just eq. (36.62).

f) The one-loop diagram is exactly the same as the vertex correction in spinor electrodynamics; in that case a photon attaches to the vertex, in the present case the neutrino current attaches to the vertex, but in both cases what gets attached does not affect the diagram. In problem 62.2 we showed that $Z_1 = 1 + O(\alpha^2)$ in Lorenz gauge, where Z_1 is the vertex renormalizing factor; hence in the present case we have $Z_G = 1 + O(\alpha^2)$ in Lorenz gauge.

g) We also have $Z_2 = 1 + O(\alpha^2)$ in Lorenz gauge. Hence $Z_G/Z_2 = 1 + O(\alpha^2)$ in Lorenz gauge (and actually in any gauge), so $c_1 = 0$. (This will change when we consider quarks, and the process of neutron decay.)

89 The Standard Model: Quark Sector

89.1) To get an allowed mass term, we must have an SU(3) singlet, which requires combining a 3 and a $\overline{3}$. However, the only 3 is also a 2 of SU(2), and both $\overline{3}$'s are singlets of SU(2), so any color-singlet combination cannot be an SU(2) singlet.

Since the Higgs field is an SU(3) singlet, to get an allowed Yukawa coupling we must again combine Weyl fields in the 3 and $\overline{3}$. Since there is only one 3, and two $\overline{3}$'s, there are just two possibilities. Then we have the option of using either the Higgs field or its hermitian conjugate. The hypercharges must sum to zero. This is true only for the two possibilities listed in eqs. (89.6–7).

Adding more fields raises the dimension to greater than four, so there are no other possible terms to consider. Q.E.D.

89.2) In problem 88.6, we showed that a vector field of mass M that couples to Dirac fields via $\mathcal{L}_{\rm int} = Z^\mu \overline{\Psi}\gamma_\mu(g_{\rm V} - g_{\rm A}\gamma_5)\Psi$, where $\overline{\Psi}$ need not be related by hermitian conjugation to Ψ, has a decay rate given by

$$\Gamma = \tfrac{1}{12\pi}(g_{\rm V}^2 + g_{\rm A}^2)M\ . \tag{89.38}$$

So we need only figure out $g_{\rm V}$ and $g_{\rm A}$ for the cases of interest.

For $W^+ \to u\overline{d}$, we see from eqs. (89.21) and (89.34) that $g_{\rm V} = g_{\rm A} = c_1 g_2/2\sqrt{2}$; using $g_2 = e/s_{\rm W}$ and $e^2 = 4\pi\alpha$, and multiplying by 3 to account for the three possible colors, we get

$$\Gamma_{W^+\to u\overline{d}} = \frac{\alpha c_1^2}{4s_{\rm W}^2}M_{\rm W}\ . \tag{89.39}$$

For $Z^0 \to \overline{u}u$, we have from eq. (89.21) and (89.24–26) that $g_{\rm V} = (\frac14 - \frac23 s_{\rm W}^2)e/s_{\rm W}c_{\rm W}$ and $g_{\rm A} = \frac14 e/s_{\rm W}c_{\rm W}$, and so

$$\Gamma_{Z^0\to\overline{u}u} = \frac{\alpha}{8s_{\rm W}^2c_{\rm W}^2}(1 - \tfrac83 s_{\rm W}^2 + \tfrac{32}{9}s_{\rm W}^4)M_{\rm Z}\ . \tag{89.40}$$

For $Z^0 \to \overline{d}d$, we have $g_{\rm V} = (-\frac14 + \frac13 s_{\rm W}^2)e/s_{\rm W}c_{\rm W}$ and $g_{\rm A} = -\frac14 e/s_{\rm W}c_{\rm W}$, and so

$$\Gamma_{Z^0\to\overline{d}d} = \frac{\alpha}{8s_{\rm W}^2c_{\rm W}^2}(1 - \tfrac43 s_{\rm W}^2 + \tfrac89 s_{\rm W}^4)M_{\rm Z}\ . \tag{89.41}$$

Putting in numbers ($\alpha = 1/127.9$, $s_{\rm W}^2 = 0.231$, $c_1 = 0.974$, $M_{\rm W} = 80.4\,{\rm GeV}$, $M_{\rm Z} = 91.2\,{\rm GeV}$), we find $\Gamma_{W^+\to u\overline{d}} = 0.645\,{\rm GeV}$, $\Gamma_{Z^0\to\overline{u}u} = 0.254\,{\rm GeV}$, and $\Gamma_{Z^0\to\overline{d}d} = 0.327\,{\rm GeV}$. To get the total width, we sum over generations. For W^+ decay, if the top quark mass could be neglected, the CKM matrix would cancel out in the sum. For $\theta_2 = \theta_3 = 0$, only the first two generations mix, and in this case the CKM matrix cancels out in the sum over $u\overline{d}$, $u\overline{s}$, $c\overline{d}$, and $c\overline{s}$. So for W^+ decay, we have three lepton generations and two quark generations, and we get

$$\Gamma_{W^+} = \frac{3\alpha}{4s_{\rm W}^2}M_{\rm W}\ . \tag{89.42}$$

For Z^0 decay, we have three generations of each of $\overline{\nu}\nu$, e^+e^-, and $\overline{d}d$, and two generations of $\overline{u}u$, for a total of

$$\Gamma_{Z^0} = \frac{\alpha}{24s_{\rm W}^2c_{\rm W}^2}(21 - 40s_{\rm W}^2 + \tfrac{160}{3}s_{\rm W}^4)M_{\rm Z}\ . \tag{89.43}$$

Putting in the numbers, we get $\Gamma_{W^+} = 2.04\,\text{GeV}$ and $\Gamma_{Z^0} = 2.44\,\text{GeV}$. These are a few per cent too low because we neglected QCD loop corrections.

89.3) The representation of the left-handed Weyl fields is three copies of $(1, 2, -\frac{1}{2}) \oplus (1, 1, +1) \oplus (3, 2, +\frac{1}{6}) \oplus (\bar{3}, 1, -\frac{2}{3}) \oplus (\bar{3}, 1, +\frac{1}{3})$. The 3–3–3 anomaly cancels if there are equal numbers of 3's and $\overline{3}$'s; in doing this counting, each SU(2) component counts separately. We see that each generation has two 3's from $(3, 2, +\frac{1}{6})$ and two $\overline{3}$'s from $(\bar{3}, 1, -\frac{2}{3}) \oplus (\bar{3}, 1, +\frac{1}{3})$; thus the 3–3–3 anomaly cancels. There is no 2–2–2 anomaly because the 2 is a pseudoreal representation. See problem 75.1 for a discussion of mixed anomalies such as 3–3–1 and 2–2–1. In general, we require $\sum_i T(\mathrm{R}_i)Q_i$ to vanish, where $T(\mathrm{R}_i)$ is the index of the representation of the nonabelian group, and Q_i is the U(1) charge. For 3–3–1, each SU(2) component counts separately. Setting $T(3) = T(\overline{3}) = 1$, we have $2(+\frac{1}{6})+(+\frac{1}{3})+(-\frac{2}{3}) = 0$. For 2–2–1, each SU(3) component counts separately. Setting $T(2) = 1$, we have $(-\frac{1}{2}) + 3(+\frac{1}{6}) = 0$. For 1–1–1, we require $\sum_i Q_i^3$ to vanish, where the sum counts each SU(2) and SU(3) component separately. We have $1\cdot 2\cdot(-\frac{1}{2})^3 + 1\cdot 1\cdot(+1)^3 + 3\cdot 2\cdot(+\frac{1}{6})^3 + 3\cdot 1\cdot(-\frac{2}{3})^3 + 3\cdot 1\cdot(+\frac{1}{3})^3 = 0$. Other possible combinations, such as 1–2–3 or 2–2–3, always involve the trace of a single SU(2) or SU(3) generator, and this vanishes. There is also a potential gravitational anomaly that is cancelled if $\sum_i Q_i$ vanishes; we have $1\cdot 2\cdot(-\frac{1}{2}) + 1\cdot 1\cdot(+1) + 3\cdot 2\cdot(+\frac{1}{6}) + 3\cdot 1\cdot(-\frac{2}{3}) + 3\cdot 1\cdot(+\frac{1}{3}) = 0$. Finally, the global SU(2) anomaly is absent if there is an even number of 2's; we have $1+3 = 4$ 2's.

89.4) See section 97.

89.5) a) These follow immediately from eqs. (36.61–62).

b) Using eq. (89.35), we see that gluon exchange would connect $\mathcal{U}$ and $\mathcal{D}$ across the γ^μ vertex; except for the group-theory factor, this is the same diagram that we had in problem 88.7, and that is the same as the vertex correction in spinor electrodynamics. This one-loop contribution to Z_C vanishes in Lorenz gauge.

c) A photon could connect $\mathcal{U}$ and $\mathcal{D}$. In this case, the one-loop contribution to Z_C vanishes in Lorenz gauge, just like the gluon contribution. A photon could connect $\mathcal{E}$ and $\mathcal{D}$. In this case, we write the interaction in the form of eq. (89.36), and make the same argument to show that the contribution to Z_C vanishes in Lorenz gauge.

d) Finally, a photon could connect $\mathcal{E}$ and $\mathcal{U}$. In this case, we write the interaction in the form of eq. (89.37), but now the vertex has a different structure. In particular, as noted in the problem, $\overline{\mathcal{E}}P_{\mathrm{R}}\mathcal{U}^{\mathrm{C}} = e^\dagger u^\dagger$. This has the same structure as $\overline{\mathcal{E}}\mathcal{E} = e^\dagger\bar{e}^\dagger + \text{h.c.}$. Thus, the one-loop diagram is the same as the one that gives the renormalizing factor for $\overline{\mathcal{E}}\mathcal{E}$ in spinor electrodynamics, namely Z_m. We must adjust the charges, though; for $e^\dagger\bar{e}^\dagger$ the charges are $+1$ and -1, while for $e^\dagger u^\dagger$ they are $+1$ and $-\frac{2}{3}$. (The problem lists the charges of the conjugate fields, whose product is of course the same.) Thus we find that Z_C is given by Z_m in spinor electrodynamics with $(+1)(-1)e^2 \to (+1)(-\frac{2}{3})e^2$. In Lorenz gauge, using the result of problem 62.2, we have $Z_C = 1 - (\frac{3}{8\pi^2})(\frac{2}{3}e^2)\varepsilon^{-1} = 1 - \frac{\alpha}{\pi}\varepsilon^{-1}$.

e) The renormalizing factor for the kinetic term of each field (Z_2 in spinor electrodynamics) vanishes in Lorenz gauge. Following the analysis of problem 88.7, we define $\mathcal{C}(\alpha,\varepsilon) \equiv \ln Z_C = \sum_{n=1}^\infty \mathcal{C}_n(\alpha)/\varepsilon^n$. Then $\gamma_C(\alpha) = \alpha\mathcal{C}_1'(\alpha) = -\frac{\alpha}{\pi}$, so $c_1 = -\frac{1}{\pi}$.

90 Electroweak Interactions of Hadrons

90.1) Eqs. (90.9–10) are in the standard form for a covariant derivative discussed in section 69. For $r_\mu = 0$, eq. (90.7) also takes this standard form. For $l_\mu = 0$, eq. (90.8) also takes this standard form. Since eq. (90.8) is the hermitian conjugate of eq. (90.7), this covers all cases.

90.2) Except for the gauge fields, this is the same as problem 83.5, and the gauge-field terms follow from straightforward matrix multiplications.

90.3) The amplitude is $\mathcal{T} = G_{\rm F} c_1 f_\pi k_\pi^\mu \overline{u}_\nu \gamma_\mu (1-\gamma_5) u_\tau$, with $k_\pi = p_\tau - p_\nu$. Using $\not{p}_\tau u_\tau = -m_\tau u_\tau$ and $\overline{u}_\nu \not{p}_\nu = 0$, we get $\mathcal{T} = -G_{\rm F} c_1 f_\pi m_\tau \overline{u}_\nu (1-\gamma_5) u_\tau$. Then $\langle |\mathcal{T}|^2 \rangle = \frac{1}{2}(G_{\rm F} c_1 f_\pi m_\tau)^2 (-8 p_\tau \cdot p_\nu)$, where the $\frac{1}{2}$ is from averaging over the initial τ spin. Next we use $-m_\pi^2 = k_\pi^2 = (p_\tau - p_\nu)^2 = -m_\tau^2 - 2p_\tau \cdot p_\nu$ to get $-2p_\tau \cdot p_\nu = m_\tau^2 - m_\pi^2$, so that $\langle |\mathcal{T}|^2 \rangle = 2(G_{\rm F} c_1 f_\pi m_\tau)^2 (m_\tau^2 - m_\pi^2)$. Then $\Gamma = |\mathbf{p}_\nu| \langle |\mathcal{T}|^2 \rangle / 8\pi m_\tau^2$, where $|\mathbf{p}_\nu| = (m_\tau^2 - m_\pi^2)/2m_\tau$, so we have

$$\Gamma_{\tau^- \to \pi^- \nu_\tau} = \frac{G_{\rm F}^2 c_1^2 f_\pi^2 m_\tau^3}{8\pi} \left(1 - \frac{m_\pi^2}{m_\tau^2}\right)^2 . \tag{90.60}$$

Putting in numbers ($m_\tau = 1.777\,\text{GeV}$), we get $\Gamma = 2.43 \times 10^{-8}\,\text{GeV}$, corresponding to a lifetime of $\hbar c / c\Gamma = (1.973 \times 10^{-11}\,\text{MeV cm})/[(2.998 \times 10^{10}\,\text{cm/s})(2.43 \times 10^{-8}\,\text{GeV})] = 2.71 \times 10^{-12}\,\text{s}$ for this mode. The measured lifetime of the τ^- for all decay modes is $2.91 \times 10^{-13}\,\text{s}$, with a branching ratio to $\pi^- \nu_\tau$ of 11.1%. Thus the lifetime for this particular mode is larger by a factor of 1/0.111, or $2.62 \times 10^{-12}\,\text{s}$. This is about 3% below our predicted value.

90.4) a) We have

$$|\mathcal{T}|^2 = \tfrac{1}{2} G_{\rm F}^2 c_1^2\, \overline{u}_p \gamma^\mu (1-g_{\rm A}\gamma_5) u_n \overline{u}_n \gamma^\nu (1-g_{\rm A}\gamma_5) u_p \overline{u}_e \gamma_\mu (1-\gamma_5) v_{\bar\nu} \overline{v}_{\bar\nu} \gamma_\nu (1-\gamma_5) u_e \; . \tag{90.61}$$

We sum over final spins and use $u_n \overline{u}_n = \frac{1}{2}(1-\gamma_5 \not{z})(-\not{p}_n + m_n)$ for the initial neutron to get

$$\begin{aligned}
\textstyle\sum_{s_p} & \overline{u}_p \gamma^\mu (1-g_{\rm A}\gamma_5) u_n \overline{u}_n \gamma^\nu (1-g_{\rm A}\gamma_5) u_p \\
&= \tfrac{1}{2} \mathrm{Tr} (-\not{p}_p + m_p) \gamma^\mu (1-g_{\rm A}\gamma_5)(1-\gamma_5 \not{z})(-\not{p}_n + m_n) \gamma^\nu (1-g_{\rm A}\gamma_5) \\
&= \tfrac{1}{2} \mathrm{Tr}\, \not{p}_p \gamma^\mu \not{p}_n \gamma^\nu (1+g_{\rm A}^2 - 2g_{\rm A}\gamma_5) \\
&\quad + \tfrac{1}{2} m_p m_n \mathrm{Tr}\, \gamma^\mu \gamma^\nu (1-g_{\rm A}^2) \\
&\quad + \tfrac{1}{2} m_p \mathrm{Tr}\, \gamma_5 \not{z} \not{p}_n \gamma^\nu \gamma^\mu (1-g_{\rm A}^2) \\
&\quad + \tfrac{1}{2} m_n \mathrm{Tr}\, \not{z} \gamma^\nu \not{p}_p \gamma^\mu [(1+g_{\rm A}^2)\gamma_5 - 2g_{\rm A}] \\
&= 2(1+g_{\rm A}^2)(p_p^\mu p_n^\nu + p_p^\nu p_n^\mu - p_p \cdot p_n\, g^{\mu\nu}) && \text{I} \\
&\quad + 4i g_{\rm A} \varepsilon^{\alpha\mu\beta\nu} p_{p\alpha} p_{n\beta} && \text{II} \\
&\quad - 2 m_p m_n (1-g_{\rm A}^2) g^{\mu\nu} && \text{III} \\
&\quad - 2i m_p (1-g_{\rm A}^2) \varepsilon^{\alpha\beta\nu\mu} z_\alpha p_{n\beta} && \text{IV} \\
&\quad - 2i m_n (1+g_{\rm A}^2) \varepsilon^{\alpha\nu\beta\mu} z_\alpha p_{p\beta} && \text{V} \\
&\quad - 4 g_{\rm A} m_n (z^\nu p_p^\mu + z^\nu p_p^\mu - z \cdot p_p\, g^{\mu\nu}) \; , && \text{VI}
\end{aligned}$$

$$
\begin{aligned}
&\textstyle\sum_{s_e,s_{\bar\nu}} \overline{u}_e\gamma_\mu(1-\gamma_5)v_{\bar\nu}\overline{v}_{\bar\nu}\gamma_\nu(1-\gamma_5)u_e \\
&\qquad = \mathrm{Tr}(-\not{p}_e+m_e)\gamma_\mu(1-\gamma_5)(-\not{p}_{\bar\nu})\gamma_\nu(1-\gamma_5) \\
&\qquad = 2\,\mathrm{Tr}\,\not{p}_e\gamma_\mu\not{p}_{\bar\nu}\,\gamma_\nu(1-\gamma_5) \\
&\qquad = 8(p_{e\mu}p_{\bar\nu\nu} + p_{e\nu}p_{\bar\nu\mu} - p_e\!\cdot\!p_{\bar\nu}\,g_{\mu\nu}) \qquad\qquad \mathrm{A} \\
&\qquad\quad + 8i\varepsilon_{\gamma\mu\delta\nu}p_e^\gamma p_{\bar\nu}^\delta\;. \qquad\qquad\qquad\qquad\quad\; \mathrm{B}
\end{aligned}
$$

We take the products of I to VI with A to B, and use $p_{n,p}\!\cdot\!p_{e,\bar\nu} \simeq -m_{n,p}E_{e,\bar\nu}$. The products that do not vanish by symmetry are

$$
\begin{aligned}
\mathrm{IA} &= +32(1+g_{\mathrm A}^2)[(p_p\!\cdot\!p_e)(p_n\!\cdot\!p_{\bar\nu}) + (p_p\!\cdot\!p_{\bar\nu})(p_n\!\cdot\!p_e)] \\
&\simeq +64(1+g_{\mathrm A}^2)m_n m_p E_e E_{\bar\nu}\;, \\
\mathrm{IIB} &= -32g_{\mathrm A}(\varepsilon^{\alpha\mu\beta\nu}\varepsilon_{\gamma\mu\delta\nu})p_{p\alpha}p_{n\beta}p_e^\gamma p_{\bar\nu}^\delta \\
&= +64g_{\mathrm A}(\delta^\alpha{}_\gamma\delta^\beta{}_\delta - \delta^\alpha{}_\delta\delta^\beta{}_\gamma)p_{p\alpha}p_{n\beta}p_e^\gamma p_{\bar\nu}^\delta \\
&= +64g_{\mathrm A}[(p_p\!\cdot\!p_e)(p_n\!\cdot\!p_{\bar\nu}) - (p_p\!\cdot\!p_{\bar\nu})(p_n\!\cdot\!p_e)] \\
&\simeq 0\;, \\
\mathrm{IIIA} &= +32m_p m_n(1-g_{\mathrm A}^2)(p_e\!\cdot\!p_{\bar\nu}) \\
&= +32m_n m_p E_e E_{\bar\nu}(1-g_{\mathrm A}^2)(\boldsymbol\beta_e\!\cdot\!\boldsymbol\beta_{\bar\nu} - 1)\;, \\
\mathrm{IVB} &= +16m_p(1-g_{\mathrm A}^2)(\varepsilon^{\alpha\beta\nu\mu}\varepsilon_{\gamma\mu\delta\nu})z_\alpha p_{n\beta}p_e^\gamma p_{\bar\nu}^\delta \\
&= -32m_p(1-g_{\mathrm A}^2)[(z\!\cdot\!p_e)(p_n\!\cdot\!p_{\bar\nu}) - (z\!\cdot\!p_{\bar\nu})(p_n\!\cdot\!p_e)] \\
&\simeq +32m_n m_p E_e E_{\bar\nu}(1-g_{\mathrm A}^2)(\hat{\mathbf z}\!\cdot\!\boldsymbol\beta_e - \hat{\mathbf z}\!\cdot\!\boldsymbol\beta_{\bar\nu})\;, \\
\mathrm{VB} &= +16m_n(1+g_{\mathrm A}^2)(\varepsilon^{\alpha\nu\beta\mu}\varepsilon_{\gamma\mu\delta\nu})z_\alpha p_{p\beta}p_e^\gamma p_{\bar\nu}^\delta \\
&= +32m_n(1+g_{\mathrm A}^2)[(z\!\cdot\!p_e)(p_p\!\cdot\!p_{\bar\nu}) - (z\!\cdot\!p_{\bar\nu})(p_p\!\cdot\!p_e)] \\
&\simeq -32m_n m_p E_e E_{\bar\nu}(1+g_{\mathrm A}^2)(\hat{\mathbf z}\!\cdot\!\boldsymbol\beta_e - \hat{\mathbf z}\!\cdot\!\boldsymbol\beta_{\bar\nu})\;, \\
\mathrm{VIA} &= -64g_{\mathrm A}m_n[(z\!\cdot\!p_e)(p_p\!\cdot\!p_{\bar\nu}) + (z\!\cdot\!p_{\bar\nu})(p_p\!\cdot\!p_e)] \\
&\simeq +64m_n m_p E_e E_{\bar\nu}\,g_{\mathrm A}(\hat{\mathbf z}\!\cdot\!\boldsymbol\beta_e + \hat{\mathbf z}\!\cdot\!\boldsymbol\beta_{\bar\nu})\;, \\
\mathrm{TOTAL} &\simeq +32m_n m_p E_e E_{\bar\nu}\Big[\,2(1+g_{\mathrm A}^2) - (1-g_{\mathrm A}^2) \\
&\qquad + (1-g_{\mathrm A}^2)\boldsymbol\beta_e\!\cdot\!\boldsymbol\beta_{\bar\nu} \\
&\qquad + (+(1-g_{\mathrm A}^2) - (1+g_{\mathrm A}^2) + 2g_{\mathrm A})\,\hat{\mathbf z}\!\cdot\!\boldsymbol\beta_e \\
&\qquad + (-(1-g_{\mathrm A}^2) + (1+g_{\mathrm A}^2) + 2g_{\mathrm A})\,\hat{\mathbf z}\!\cdot\!\boldsymbol\beta_{\bar\nu}\,\Big]\;,
\end{aligned}
$$

where $\boldsymbol\beta_i = \mathbf{p}_i/E_i$; $\langle|\mathcal{T}|^2\rangle = \frac{1}{2}G_{\mathrm F}^2c_1^2 \times \mathrm{TOTAL}$, which agrees with eqs. (90.43–44).

b) We have

$$
\Gamma = \frac{1}{2m_n}\int \widetilde{dp}_p\,\widetilde{dp}_e\,\widetilde{dp}_{\bar\nu}\,(2\pi)^4\delta^4(p_n-p_p-p_e-p_{\bar\nu})\langle|\mathcal{T}|^2\rangle\;. \tag{90.62}
$$

The correlation terms integrate to zero. We then have

$$
\begin{aligned}
\Gamma &= \frac{G_{\mathrm F}^2c_1^2}{(2\pi)^5}(1+3g_{\mathrm A}^2)\int d^3p_p\,d^3p_e\,d^3p_{\bar\nu}\,\delta^4(p_n-p_p-p_e-p_{\bar\nu}) \\
&= \frac{G_{\mathrm F}^2c_1^2}{(2\pi)^5}(1+3g_{\mathrm A}^2)\int d^3p_e\,d^3p_{\bar\nu}\,\delta(m_n-m_p-E_e-E_{\bar\nu})\;,
\end{aligned} \tag{90.63}
$$

where we set $E_p = m_p$ since $E_p - m_p \simeq \mathbf{p}_p^2/2m_p \ll E_e + E_{\bar\nu}$. Now using $d^3p_e = 4\pi p_e E_e\,dE_e$ and $d^3p_{\bar\nu} = 4\pi E_{\bar\nu}^2\,dE_{\bar\nu}$, and setting $\Delta = m_n - m_p = 1.293\,\text{MeV}$ and $r = m_e/\Delta = 0.3952$, we get

$$\begin{aligned}\Gamma &= \frac{G_\text{F}^2 c_1^2}{2\pi^3}(1{+}3g_\text{A}^2)\int p_e E_e\,dE_e\,E_{\bar\nu}^2\,dE_{\bar\nu}\,\delta(m_n{-}m_p{-}E_e{-}E_{\bar\nu}) \\ &= \frac{G_\text{F}^2 c_1^2}{2\pi^3}(1{+}3g_\text{A}^2)\int_{m_e}^{\Delta}(E_e^2{-}m_e^2)^{1/2}E_e(\Delta{-}E_e)^2\,dE_e \\ &= \frac{G_\text{F}^2 c_1^2}{60\pi^3}(1{+}3g_\text{A}^2)\Delta^5 f(r)\,,\end{aligned}$$

where

$$f(r) = (1{-}\tfrac{9}{2}r^2{-}4r^4)(1{-}r^2)^{1/2} + \tfrac{15}{2}r^4\ln[r^{-1}{+}(r^{-2}{-}1)^{1/2}]\,,$$

and $f(0.3952) = 0.4724$. Thus we find $\Gamma = (1.184\times10^{-25}\,\text{MeV})(1{+}3g_\text{A}^2)$. Comparing with $\Gamma = \hbar c/c\tau = (1.973\times10^{-11}\,\text{MeV cm})/[(2.998\times10^{10}\text{cm/s})(885.7\,\text{s})] = 7.430\times10^{-25}\,\text{MeV}$, we find $g_\text{A} = 1.326$, about 4% higher than the actual value of $g_\text{A} = 1.27$.

90.5) From eq. (88.48), we see that $G_\text{F} = G_\text{F}(M_\text{W})$ should be replaced by $[1 - c_1\alpha\ln(M_\text{W}/\mu)]G_\text{F}$, where c_1 (which is not the cosine of the Cabibbo angle!) was computed to be $c_1 = -\frac{1}{\pi}$ for the interaction that leads to neutron decay. The scale μ should be taken to be a typical energy in the relevant process, in this case $\mu \sim m_p$. Since the neutron decay rate depends on G_F^2, the enhancement factor is $[\ldots]^2 = 1 + \frac{2}{\pi}\alpha\ln(M_\text{W}/m_p) \simeq 1.021$. Thus the computed value of $1{+}3g_\text{A}^2$ is now smaller by 1/1.021, and we get $g_\text{A} = 1.310$, which is an improvement, but still too large by 3%.

90.6) We have

$$\mathcal{T} = 2\sqrt{2}G_\text{F}\Big(\tfrac{1}{\sqrt{2}}c_1(k_+{+}k_0)^\mu\Big)\Big(\tfrac{1}{2}\bar u_e\gamma_\mu(1{-}\gamma_5)v_{\bar\nu}\Big)\,, \tag{90.64}$$

and so

$$\begin{aligned}\langle|\mathcal{T}|^2\rangle &= c_1^2G_\text{F}^2\,\text{Tr}\,(\not k_+{+}\not k_0)(1{-}\gamma_5)(-\not p_{\bar\nu})(\not k_+{+}\not k_0)(1{-}\gamma_5)(-\not p_e{+}m_e) \\ &= 2c_1^2G_\text{F}^2\,\text{Tr}(\not k_+{+}\not k_0)(-\not p_{\bar\nu})(\not k_+{+}\not k_0)(1{-}\gamma_5)(-\not p_e) \\ &= 8c_1^2G_\text{F}^2[2(k_+{+}k_0){\cdot}p_{\bar\nu}(k_+{+}k_0){\cdot}p_e - (k_+{+}k_0)^2p_{\bar\nu}{\cdot}p_e]\,.\end{aligned} \tag{90.65}$$

We have $k_+\cdot p_{\bar\nu} = -m_+E_{\bar\nu}$ and $k_+\cdot p_e = -m_+E_e$, and, since the π^0 is nonrelativistic, $(k_+{+}k_0)^2 \simeq -4m_+^2$, $k_0{\cdot}p_{\bar\nu} \simeq -m_0E_{\bar\nu}$, and $k_0{\cdot}p_e \simeq -m_0E_e$. Thus

$$\langle|\mathcal{T}|^2\rangle = 8c_1^2G_\text{F}^2\Big[8m_+^2E_{\bar\nu}E_e + 4m_+^2(\mathbf{p}_{\bar\nu}{\cdot}\mathbf{p}_e - E_{\bar\nu}E_e)\Big]\,. \tag{90.66}$$

The $\mathbf{p}_{\bar\nu}{\cdot}\mathbf{p}_e$ term will integrate to zero, so we have

$$\langle|\mathcal{T}|^2\rangle \to 32c_1^2G_\text{F}^2m_+^2E_{\bar\nu}E_e\,. \tag{90.67}$$

From here it is exactly the same as neutron decay, and so we get

$$\begin{aligned}\Gamma &= \frac{G_\text{F}^2c_1^2}{30\pi^3}\Delta^5 f(r) \\ &= (2.815\times10^{-22}\,\text{MeV})c_1^2\,,\end{aligned} \tag{90.68}$$

where $\Delta = m_+ - m_0 = 4.594\,\text{MeV}$, $r = m_e/\Delta = 0.1112$, and $f(r)$ was defined in the solution to problem 90.4. Comparing to $\Gamma = [(1.973\times 10^{-11}\,\text{MeV cm})/(2.998\times 10^{10}\,\text{cm/s})](0.3972\,\text{s}^{-1}) = 2.614\times 10^{-22}\,\text{MeV}$ yields $c_1 = 0.9634$, about 1% too low.

90.7) We have $|\mathcal{T}|^2 = (\alpha/\pi f_\pi)^2 \varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta} k_{1\mu}k_{2\rho}k_{1\alpha}k_{2\gamma}\varepsilon_{1\nu}\varepsilon_{2\sigma}\varepsilon^*_{1\beta}\varepsilon^*_{2\delta}$. Summing over the photon polarizations yields

$$\begin{aligned}\langle|\mathcal{T}|^2\rangle &= (\alpha/\pi f_\pi)^2 \varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\alpha\nu\gamma\sigma} k_{1\mu}k_{2\rho}k_1^\alpha k_2^\gamma \\ &= (\alpha/\pi f_\pi)^2 (2\delta^\mu{}_\gamma\delta^\rho{}_\alpha - 2\delta^\mu{}_\alpha\delta^\rho{}_\gamma) k_{1\mu}k_{2\rho}k_1^\alpha k_2^\gamma \\ &= (\alpha/\pi f_\pi)^2 [2(k_1\cdot k_2)^2 - 0] \ . \end{aligned} \tag{90.69}$$

Using $-m_\pi^2 = (k_1+k_2)^2 = 2k_1\cdot k_2$, we get

$$\langle|\mathcal{T}|^2\rangle = \frac{\alpha^2 m_\pi^4}{2\pi^2 f_\pi^2} \ . \tag{90.70}$$

To get a rate for outgoing identical (symmetry factor $S=2$) massless particles, we divide by $32\pi m_\pi$, which yields

$$\Gamma = \frac{\alpha^2 m_\pi^3}{64\pi^3 f_\pi^2} \ , \tag{90.71}$$

in agreement with eq. (90.59). Putting in numbers, we get $\Gamma = 7.7\,\text{eV}$. The measured lifetime (with a 7% uncertainty) is $8.4\times 10^{-17}\,\text{s}$, corresponding to $\Gamma = 7.8\,\text{eV}$. (The branching ratio for this mode is 98.8% with the remaining 1.2% almost entirely $e^+e^-\gamma$.)

91 Neutrino Masses

91.1) The symmetry that gives rise to lepton number conservation is $\ell \to e^{-i\alpha}\ell$, $\overline{e} \to e^{+i\alpha}\overline{e}$. In order for $\mathcal{L}_{\nu\,\mathrm{Yuk}}$ to be invariant, we must take $\bar{\nu} \to e^{+i\alpha}\bar{\nu}$ as well. But then $\mathcal{L}_{\bar{\nu}\,\mathrm{mass}}$ is not invariant. This leads to processes such as $\mu^- \to e^-\gamma$, but the rate is unobservably low; see *Cheng & Li* for a detailed calculation.

92 Solitons and Monopoles

92.1) a) Let $\mathbf{y} = \mathbf{x}/\alpha$; then $\int d^D x\, V(\varphi_i(\mathbf{x}/\alpha)) = \alpha^D \int d^D y\, V(\varphi_i(\mathbf{y})) = \alpha^D U$. Also, $\nabla_x \varphi_i(\mathbf{x}/\alpha) = \alpha^{-1}\nabla_y \varphi_i(\mathbf{y})$, and so $\int d^D x\,(\nabla_x \varphi_i(\mathbf{x}/\alpha))^2 = \alpha^{D-2}\int d^D y\,(\nabla_y \varphi_i(\mathbf{y}))^2 = \alpha^{D-2}T$.

b) The energy as a function of α is $E(\alpha) = \alpha^{D-2}T + \alpha^D U$. The energy is supposed to be minimized by the original solution, with $\alpha = 1$, and hence $E'(1) = 0$.

c) $E'(\alpha) = (D-2)\alpha^{D-3}T + D\alpha^{D-1}U$, so $E'(1) = (D-2)T + DU$. Since T and U are both positive-definite, $E'(1)$ cannot vanish for $D \geq 2$.

92.2) a) $U^\dagger U = 1$, so $\delta U^\dagger U + U^\dagger \delta U = 0$. Multiplying by $U^\dagger$, we get $\delta U^\dagger = -U^{\dagger 2}\delta U$.

b) We have

$$
\begin{aligned}
\delta(U\partial_\phi U^\dagger) &= \delta U \partial_\phi U^\dagger + U\partial_\phi \delta U^\dagger \\
&= \delta U \partial_\phi U^\dagger + U\partial_\phi(-U^{\dagger 2}\delta U) \\
&= \delta U \partial_\phi U^\dagger + U[-2U^\dagger(\partial_\phi U^\dagger)\delta U - U^{\dagger 2}\partial_\phi \delta U] \\
&= \delta U \partial_\phi U^\dagger + [-2(\partial_\phi U^\dagger)\delta U - U^\dagger \partial_\phi \delta U] \\
&= -(\partial_\phi U^\dagger)\delta U - U^\dagger \partial_\phi \delta U \\
&= -\partial_\phi(U^\dagger \delta U)\ .
\end{aligned} \tag{92.64}
$$

c) $\delta n = \frac{i}{2\pi}\int_0^{2\pi} d\phi\, \partial_\phi(U^\dagger \delta U) = \frac{i}{2\pi}U^\dagger \delta U\big|_{\phi=0}^{\phi=2\pi} = 0$, since U is continuous and $\phi = 0$ is identified with $\phi = 2\pi$.

92.3) Deform $U_n(\phi)$ so that it equals one for $0 \leq \phi \leq \pi$, and deform $U_k(\phi)$ so that it equals one for $\pi \leq \phi \leq 2\pi$. The winding number for U_n is then given by $\frac{i}{2\pi}\int_0^\pi d\phi\, U_n \partial_\phi U_n^\dagger$, and the winding number for U_k by $\frac{i}{2\pi}\int_\pi^{2\pi} d\phi\, U_k \partial_\phi U_k^\dagger$, since the regions where $U = 1$ have $\partial_\phi U^\dagger = 0$, and hence do not contribute to the integral. For the deformed U's, $U_n U_k = U_k$ for $0 \leq \phi \leq \pi$, and $U_n U_k = U_n$ for $\pi \leq \phi \leq 2\pi$. Hence, in doing the winding-number integral for $U_n U_k$, we get the winding number of U_k from $0 \leq \phi \leq \pi$, plus the winding number of U_n from $\pi \leq \phi \leq 2\pi$. Q.E.D.

92.4) For $\rho \ll 1$, a and f are small, and we can neglect them compared to 1. Eq. (92.30) then becomes $f'' + f'/\rho - n^2 f/\rho^2$. Plugging in the ansatz $f \sim \rho^\nu$, we find $(\nu^2 - n^2)\rho^{\nu-2}$, and hence $\nu = n$ (since $\nu = -n$ does not satisfy the boundary condition that f vanish at $\rho = 0$). Eq. (92.31) then becomes $a'' - a'/\rho + f^2 = 0$. Plugging in the ansatz $a \sim \rho^\alpha$, we find $(\alpha^2 - 2\alpha)\rho^{\alpha-2} + \rho^{2n}$. The first term dominates for $\alpha < 2n{+}2$; in this case, we require the coefficient to vanish, and hence $\alpha = 2$ (since $\alpha = 0$ does not satisfy the boundary condition that a vanish at $\rho = 0$). For $\alpha = 2$, the second term is subdominant for any nonzero n.

For $\rho \gg 1$, let $a = 1 - A$ and $f = 1 - F$, with A and F both $\ll 1$. Then, for $\rho \gg 1$, eq. (92.32) becomes $-A'' + A = 0$; the solution that vanishes as $\rho \to \infty$ is $A \sim e^{-\rho}$. Eq. (92.30) becomes $-F'' + \beta^2 F = 0$; the solution that vanishes as $\rho \to \infty$ is $F \sim e^{-\beta\rho}$. However, if $\beta > 2$ then actually it is the third term in eq. (92.30) that dominates at large ρ, since $(1-a)^2 = A^2 \sim e^{-2\rho}$ while the remaining terms go like $e^{-\beta\rho}$. Hence, for $\beta > 2$, we must have $F \sim e^{-2\rho}$ to achieve appropriate cancellations at large ρ.

92.5) We have

$$\begin{aligned}
\hat\varphi &= (\sin\theta\cos n\phi,\ \sin\theta\sin n\phi,\ \cos\theta)\ , \\
\partial_\theta\hat\varphi &= (\cos\theta\cos n\phi,\ \cos\theta\sin n\phi,\ -\sin\theta)\ , \\
\partial_\phi\hat\varphi &= (-n\sin\theta\sin n\phi,\ n\sin\theta\cos n\phi,\ 0)\ ,
\end{aligned} \tag{92.65}$$

and hence

$$\begin{aligned}
\varepsilon^{abc}\hat\varphi^a\partial_\theta\hat\varphi^b\partial_\phi\hat\varphi^c &= \begin{vmatrix} \sin\theta\cos n\phi & \sin\theta\sin n\phi & \cos\theta \\ \cos\theta\cos n\phi & \cos\theta\sin n\phi & -\sin\theta \\ -n\sin\theta\sin n\phi & n\sin\theta\cos n\phi & 0 \end{vmatrix} \\
&= n\sin\theta\ .
\end{aligned} \tag{92.66}$$

Also, $\varepsilon^{ij}\varepsilon^{abc}\hat\varphi^a\partial_i\hat\varphi^b\partial_j\hat\varphi^c = 2\varepsilon^{abc}\hat\varphi^a\partial_\theta\hat\varphi^b\partial_\phi\hat\varphi^c = 2n\sin\theta$. Thus the right-hand side of eq. (92.35) becomes $\frac{n}{4\pi}\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\,\sin\theta = n$.

92.6) a) Since $\hat\varphi\cdot\hat\varphi = 1$, we have both $\delta(\hat\varphi\cdot\hat\varphi) = 2\hat\varphi\cdot\delta\hat\varphi = 0$ and $\partial_i(\hat\varphi\cdot\hat\varphi) = 2\hat\varphi\cdot\partial_i\hat\varphi = 0$.

b) Since $\delta\hat\varphi$, $\partial_1\hat\varphi$, and $\partial_2\hat\varphi$ are orthogonal to $\hat\varphi$, they lie in a plane, and so $(\partial_1\hat\varphi\times\partial_2\hat\varphi)\cdot\delta\hat\varphi = 0$; equivalently, $\varepsilon^{abc}\delta\hat\varphi^a\partial_1\hat\varphi^b\partial_2\hat\varphi^c = 0$. We can replace 1 and 2 with i and j, since this expression is trivially zero for $i = j$. Q.E.D.

c) We have

$$\delta(\hat\varphi^a\partial_i\hat\varphi^b\partial_j\hat\varphi^c) = (\delta\hat\varphi^a)\partial_i\hat\varphi^b\partial_j\hat\varphi^c + \hat\varphi^a(\partial_i\delta\hat\varphi^b)\partial_j\hat\varphi^c + \hat\varphi^a\partial_i\hat\varphi^b(\partial_j\delta\hat\varphi^c)\ . \tag{92.67}$$

We use this in eq. (92.35) to get δn. When contracted with ε^{abc}, the first term on the right-hand side of eq. (92.67) vanishes by our result in part (b). In the second term, we integrate ∂_i by parts to get

$$\hat\varphi^a(\partial_i\delta\hat\varphi^b)\partial_j\hat\varphi^c \to -(\partial_i\hat\varphi^a)\delta\hat\varphi^b\partial_j\hat\varphi^c - \hat\varphi^a\delta\hat\varphi^b\partial_i\partial_j\hat\varphi^c\ . \tag{92.68}$$

When contracted with ε^{abc}, the first term on the right-hand side of eq. (92.68) vanishes by our result in part (b). When contracted with ε^{ij}, the second term on the right-hand side vanishes because $\varepsilon^{ij}\partial_i\partial_j = 0$. A similar analysis applies to the third term in eq. (92.67). We conclude that $\delta n = 0$. Q.E.D.

93 Instantons and Theta Vacua

93.1) Let $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $U^\dagger = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$ and $U^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. We also have $\det U = ad - bc$, so imposing $\det U = 1$ we can write $U^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Then $U^\dagger = U^{-1}$ yields $d = a^*$ and $c = -b^*$. Let $a = a_4 + ia_3$ and $b = a_2 + ia_1$ with a_μ real, so that we now have $U = \begin{pmatrix} a_4 + ia_3 & i(a_1 - ia_2) \\ i(a_1 + ia_2) & a_4 - ia_3 \end{pmatrix} = a_4 + i\vec{a}\cdot\vec{\sigma}$, with $\det U = a_\mu a_\mu = 1$.

93.2) $\langle n'|H|\theta\rangle = \sum_n e^{-in\theta}\langle n'|H|n\rangle = \sum_n e^{-in\theta} f(n'-n)$. Replace the dummy summation variable n with $m + n'$; then $\langle n'|H|\theta\rangle = \sum_m e^{-i(m+n')\theta} f(-m) = e^{-in'\theta}\sum_m e^{-im\theta} f(-m) = \langle n'|\theta\rangle E_\theta$, where $E_\theta = \sum_m e^{-im\theta} f(-m)$ is the energy eigenvalue.

93.3) a) $U^\dagger U = 1$ implies $\delta(U^\dagger U) = 0$, and so $\delta U^\dagger U + U^\dagger \delta U = 0$. Mulitply on the right by $U^\dagger$ and solve for $\delta U^\dagger = -U^\dagger \delta U U^\dagger$.

$$\begin{aligned} \delta(U\partial_k U^\dagger) &= \delta U \partial_k U^\dagger + U\partial_k \delta U^\dagger \\ &= \delta U \partial_k U^\dagger - U\partial_k (U^\dagger \delta U U^\dagger) \\ &= \delta U \partial_k U^\dagger - U\partial_k U^\dagger \delta U U^\dagger - UU^\dagger \partial_k \delta U U^\dagger - UU^\dagger \delta U \partial_k U^\dagger \\ &= -U\partial_k U^\dagger \delta U U^\dagger - UU^\dagger \partial_k \delta U U^\dagger \\ &= -U(\partial_k U^\dagger \delta U + U^\dagger \partial_k \delta U)U^\dagger \\ &= -U\partial_k (U^\dagger \delta U)U^\dagger \ . \end{aligned} \tag{93.46}$$

b) The variations of $U\partial_i U^\dagger$, $U\partial_j U^\dagger$, and $U\partial_k U^\dagger$ contribute equally to δn after cyclic permutations of the trace. We have

$$\begin{aligned} &\varepsilon^{ijk}\,\mathrm{Tr}[(U\partial_i U^\dagger)(U\partial_j U^\dagger)\delta(U\partial_k U^\dagger)] \\ &\quad = -\varepsilon^{ijk}\,\mathrm{Tr}[(U\partial_i U^\dagger)(U\partial_j U^\dagger)U\partial_k(U^\dagger \delta U)U^\dagger] \\ &\quad = -\varepsilon^{ijk}\,\mathrm{Tr}[\partial_i U^\dagger U \partial_j U^\dagger U \partial_k (U^\dagger \delta U)] \ . \end{aligned} \tag{93.47}$$

We used the cyclic property of the trace and $U^\dagger U = 1$ to get the last line. After integrating ∂_k by parts, terms with two derivatives acting on a single $U^\dagger$ vanish when contracted with ε^{ijk}. The remaining terms are

$$\begin{aligned} &-\varepsilon^{ijk}\,\mathrm{Tr}[\partial_i U^\dagger U \partial_j U^\dagger U \partial_k (U^\dagger \delta U)] \\ &\quad = +\,\varepsilon^{ijk}(\mathrm{Tr}[\partial_i U^\dagger \partial_k U \partial_j U^\dagger \delta U] + \mathrm{Tr}[\partial_i U^\dagger U \partial_j U^\dagger \partial_k U U^\dagger \delta U]) \ , \end{aligned} \tag{93.48}$$

where we used $UU^\dagger = 1$ in the first term. In the second term, we now use $U\partial_j U^\dagger = -\partial_j U U^\dagger$ and $\partial_k U U^\dagger = -U\partial_k U^\dagger$, followed by $U^\dagger U = 1$, to get

$$\begin{aligned} &-\varepsilon^{ijk}\,\mathrm{Tr}[\partial_i U^\dagger U \partial_j U^\dagger U \partial_k (U^\dagger \delta U)] \\ &\quad = +\,\varepsilon^{ijk}(\mathrm{Tr}[\partial_i U^\dagger \partial_k U \partial_j U^\dagger \delta U] + \mathrm{Tr}[\partial_i U^\dagger \partial_j U \partial_k U^\dagger \delta U]) \ . \end{aligned} \tag{93.49}$$

The two terms are now symmetric on $j \leftrightarrow k$, and so cancel when contracted with ε^{ijk}.

93.4) The argument is identical to the one given for problem 92.3, with three-dimensional space (plus a point at infinity to get S^3) replacing the circle S^1.

93.5) This is just plug in and grind, best done with a symbolic manipulation program like Mathemetica. You should find that $(U\partial_\chi U^\dagger)(U\partial_\psi U^\dagger)(U\partial_\phi U^\dagger) = -(U\partial_\psi U^\dagger)(U\partial_\chi U^\dagger)(U\partial_\phi U^\dagger) = -n(\sin^2\chi\sin\psi)I$, which just provides the measure for the 3-sphere. Of course the final result is $n = n$.

93.6) For a unit vector $\hat{n}$, $(\hat{n}\cdot\vec{\sigma})^2 = I$, and so $\exp[i\chi\hat{n}\cdot\vec{\sigma}] = (\cos\chi) + i(\sin\chi)\hat{n}\cdot\vec{\sigma}$. The right-hand side of eq. (93.29) takes this form, with $\hat{n} = (\sin\psi\cos\phi, \sin\psi\sin\phi, \cos\psi)$; hence $U = \exp[i\chi\hat{n}\cdot\vec{\sigma}]$, and $U^n = \exp[in\chi\hat{n}\cdot\vec{\sigma}]$, which is the same as U with $\chi \to n\chi$. Defining $U_n \equiv U^n$, we then find $(U_n\partial_\chi U_n^\dagger)(U_n\partial_\psi U_n^\dagger)(U_n\partial_\phi U_n^\dagger) = -(U_n\partial_\psi U_n^\dagger)(U_n\partial_\chi U_n^\dagger)(U_n\partial_\phi U_n^\dagger) = -n(\sin^2 n\chi\sin\psi)I$. Since $\int_0^\pi d\chi\,\sin^2 n\chi = \int_0^\pi d\chi\,\sin^2\chi$, the final result is the same as for problem 93.5, namely $n = n$, which of course is in accord with the theorem of problem 93.4.

94 Quarks and Theta Vacua

94.1) To simplify the notation let us define $\bar{m} \equiv (c_{-}{+}4c_4)\tilde{m}$. Then the mass terms are

$$\begin{aligned} -\overline{\mathcal{N}}(m_N + i\theta\bar{m}\gamma_5)\mathcal{N} &= -m_N\overline{\mathcal{N}}(1 + i\theta\bar{m}\gamma_5/m_N)\mathcal{N} \\ &\simeq -m_N\overline{\mathcal{N}}\exp(i\theta\bar{m}\gamma_5/m_N)\mathcal{N}\;, \end{aligned} \tag{94.40}$$

where the last approximate equality holds up to terms of order $\theta^2\bar{m}^2/m_N^2$. If we now make the field redefinition $\mathcal{N} \to e^{-i\alpha\gamma_5}\mathcal{N}$ (which implies $\overline{\mathcal{N}} \to \overline{\mathcal{N}}e^{-i\alpha\gamma_5}$) with $\alpha = \frac{1}{2}\theta\bar{m}/m_N$, eq. (94.40) becomes simply $-m_N\overline{\mathcal{N}}\mathcal{N}$. Next we note that $\overline{\mathcal{N}}\gamma^\mu\mathcal{N} \to \overline{\mathcal{N}}e^{-i\alpha\gamma_5}\gamma^\mu e^{-i\alpha\gamma_5}\mathcal{N} = \overline{\mathcal{N}}\gamma^\mu e^{+i\alpha\gamma_5}e^{-i\alpha\gamma_5}\mathcal{N} = \overline{\mathcal{N}}\gamma^\mu\mathcal{N}$, so any term with γ^μ or $\gamma^\mu\gamma_5$ sandwiched between $\overline{\mathcal{N}}$ and $\mathcal{N}$ is left unchanged by this transformation. All the $\mathcal{N}$-dependent terms in $\mathcal{L}$ that are not of this form already have a factor of a quark mass, so the transformation $\mathcal{N} \to e^{-i\alpha\gamma_5}\mathcal{N} \simeq (1{-}i\alpha\gamma_5)\mathcal{N}$ with $\alpha = O(m)$ will yield only terms with at least two powers of a quark mass.

94.2) a) The new Yukawa coupling is obviously invariant, and all other terms involve both the fields and their hermitian conjugates, and so are also invariant.

b) If we define a Dirac field $\Psi = \begin{pmatrix} \chi \\ \xi^\dagger \end{pmatrix}$, then the PQ transformation is $\Psi \to e^{-i\alpha\gamma_5}\Psi$, which, as we have seen, changes θ to $\theta + 2\alpha$.

c) $y\Phi\chi\xi + \text{h.c.} \to \frac{1}{\sqrt{2}}yf(\chi\xi + \chi^\dagger\xi^\dagger) = m\overline{\Psi}\Psi$, with $m = \frac{1}{\sqrt{2}}yf$; this has the wrong sign, but this is fixed by making the transformation of part (b) with $\alpha = \frac{1}{2}\pi$.

d) Without the effects of the anomaly, the a field would be a massless Goldstone boson, and so is part of the low-energy theory. It gets a standard kinetic term from the kinetic term for Φ. Since a PQ transformation changes θ by $+2\alpha$ and the phase of Φ (which is a/f) by -2α, $\theta + a/f$ is invariant, and is the variable that should appear in the low-energy theory.

e) The potential is given by eq. (94.13) with $\theta \to \theta + a/f$; since a is a field its value must be chosen to minimize the energy. The minimum occurs when the argument of each cosine is zero, and this is achieved for $\phi = 0$, corresponding to $U = I$, and $a = -f\theta$.

f) If we substitute eq. (94.14) for ϕ back into eq. (94.13) and expand in powers of θ, we get $V = -2v^3(m_u{+}m_d) + \bar{m}v^3\theta^2 + \ldots$. We now replace θ with $\theta + a/f$; then the minimum is at $a = -f\theta$, so we write $a = -f\theta + \tilde{a}$. Dropping the constant term, the potential becomes $V = \frac{1}{2}(2\bar{m}v^3/f^2)\tilde{a}^2$, and we see that $m_a^2 = (2\bar{m}v^3/f^2)$. Using $m_\pi^2 = 2(m_u{+}m_d)v^3/f_\pi^2$, we get $m_a^2 = [\bar{m}/(m_u{+}m_d)](f_\pi/f)^2 m_\pi^2$.

Alternatively, we can start with eq. (94.13), set $a = -f\theta + \tilde{a}$ and $\phi = \pi^0/f_\pi$, find the mass-squared matrix for π^0 and $\tilde{a}$, and diagonalize it in the limit $f \gg f_\pi$. The two resulting eigenvalues are m_π^2 and the value of m_a^2 that we just computed. This method gives the right answer even if f is not much larger than f_π.

g) Since a appears in the low-energy lagrangian via the replacement $\theta \to \theta + a/f$, or equivalently $\theta \to \tilde{a}/f$, it is obvious that $\tilde{a}$ is always accompanied (except in its kinetic term, which has a different origin) by a factor of $1/f$. Thus any interaction carries this suppression.

95 Supersymmetry

95.1) Consider $\sum_A[\{Q_{1A},Q^\dagger_{\dot{1}A}\}+\{Q_{2A},Q^\dagger_{\dot{2}A}\}]$. Using eq. (92.6), we have

$$\begin{aligned}\textstyle\sum_A[\{Q_{1A},Q^\dagger{}_{\dot{1}A}\}+\{Q_{2A},Q^\dagger{}_{\dot{2}A}\}] &= -2\mathcal{N}(\sigma^\mu_{1\dot{1}}+\sigma^\mu_{2\dot{2}})P_\mu \\ &= -2\mathcal{N}(2\delta_0{}^\mu)P_\mu \\ &= -4\mathcal{N}P_0 \\ &= +4\mathcal{N}H\ . \end{aligned} \tag{95.81}$$

Since $Q_{1A}Q^\dagger{}_{\dot{1}A}$, $Q^\dagger{}_{\dot{1}A}Q_{1A}$, $Q_{2A}Q^\dagger{}_{\dot{2}A}$ and $Q^\dagger{}_{\dot{2}A}Q_{2A}$ are all positive operators, the eigenvalues of H must be nonnegative. A state $|0\rangle$ with $H|0\rangle=0$ must also obey $Q_{aA}|0\rangle=0$ and $Q^\dagger{}_{\dot{a}A}|0\rangle=0$, because any nonzero state $|\psi\rangle$ as a result would lead to $\langle 0|H|0\rangle\geq\langle\psi|\psi\rangle>0$.

95.2) a) We have $\langle 0|\{\psi_c,Q_a\}|0\rangle=-i\sqrt{2}\varepsilon_{ac}\langle 0|F|0\rangle$. If $\langle 0|F|0\rangle\neq 0$, then $\langle 0|(Q_a\psi_c+\psi_cQ_a)|0\rangle\neq 0$, and hence either $Q_a|0\rangle\neq 0$ or $\langle 0|Q_a\neq 0$; the latter implies (by hermitian conjugation) that $Q^\dagger_{\dot{a}}|0\rangle\neq 0$. Thus if $\langle 0|F|0\rangle\neq 0$, either $Q_a|0\rangle\neq 0$ or $Q^\dagger_{\dot{a}}|0\rangle\neq 0$. Thus the vacuum is not annihilated by at least one supercharge, and so supersymmetry is spontaneously broken.

b) We have $[V,Q_a]=-i\partial_aV+\sigma^\mu_{a\dot{c}}\theta^{*\dot{c}}\partial_\mu V$. The relevant term in $[V,Q_a]$ is $\theta^*\theta^*\theta^c\{\lambda_c,Q_a\}$. We can get a term with this theta structure either from $-i\partial_a$ acting on $\frac{1}{2}\theta\theta\theta^*\theta^*D$, or from $\sigma^\mu_{a\dot{c}}\theta^{*\dot{c}}\partial_\mu$ acting on $\theta\sigma^\nu\theta^*v_\nu$. The latter is a mess, so much so that it is best to redefine that components of V, replacing D with $D+\frac{1}{2}\partial^2C$ and λ with $\lambda+\frac{1}{2}i\sigma^\mu\partial_\mu\chi^\dagger$ (minus signs not guaranteed); then we get simpler transformation rules for the component fields. (See *Wess and Bagger* or *Weinberg III* for more details.) Since the vacuum is Lorentz invariant, we have $\langle 0|\partial_\nu v_\mu|0\rangle=0$, and hence $\langle 0|\{\lambda_c,Q_a\}|0\rangle=-i\varepsilon_{ac}\langle 0|D|0\rangle$. Following the argument from part (a), we conclude that $\langle 0|D|0\rangle\neq 0$ results in the spontaneous breaking of supersymmetry.

95.3) a) We have $V=|\partial W/\partial A|^2+|\partial W/\partial B|^2+|\partial W/\partial C|^2$. From eq. (94.45) we have $F_i=-(\partial W/\partial A_i)^\dagger$, so if every $F_i=0$, then every $\partial W/\partial A_i=0$, and hence $V=0$. We have

$$\begin{aligned}\partial W/\partial A &= \kappa(C^2-v^2)\ , \\ \partial W/\partial B &= mC\ , \qquad (95.82)\\ \partial W/\partial C &= mB+2\kappa AC\ . \qquad (95.83)\end{aligned}$$

However, we cannot have both $\partial W/\partial A=0$ and $\partial W/\partial B=0$, since the former requires $C=v$ and the latter $C=0$. Instead, we must minimize $|\partial W/\partial A|^2+|\partial W/\partial B|^2$, which yields $2\kappa C^\dagger(C^2-v^2)+m^2C=0$; the solution is $C=\pm(v^2-m^2/2\kappa^2)^{1/2}$. (We can always choose conventions so that the sign is positive.) Then we have, at the minimum, $\partial W/\partial A=-\frac{1}{2}m^2$ and $\partial W/\partial B=m\langle C\rangle$, where $\langle C\rangle\equiv(v^2-m^2/2\kappa^2)^{1/2}$. Both of these are nonzero (unless v^2 happens to equal $m^2/2\kappa^2$ exactly). For simplicity, we take κ, v, and m to be real, and we also assume that $v^2>m^2/2\kappa^2$, so that $\langle C\rangle$ is real.

b) $|\partial W/\partial A|^2+|\partial W/\partial B|^2$ is minimized for $C=\langle C\rangle$. Minimizing $|\partial W/\partial C|^2$ then yields $\partial W/\partial C=0$, which fixes $mB+2\kappa\langle C\rangle A=0$. Thus we have $\langle B\rangle=-2\kappa\langle C\rangle\langle A\rangle/m$, but $\langle A\rangle$ is arbitrary. This will lead to a massless (complex) scalar field corresponding to the flat direction.

To see this explicitly, define a mixing angle $\theta \equiv \tan^{-1}(2\kappa\langle C\rangle/m)$, and new fields

$$\begin{aligned} X &= (\cos\theta)A - (\sin\theta)B \ , \\ Y &= (\sin\theta)A + (\cos\theta)B \ . \end{aligned} \tag{95.84}$$

Then $\langle Y\rangle = 0$ and $\langle X\rangle = (4\kappa^2 v^2/m^2 - 1)^{1/2}\langle A\rangle$. We then find $\partial W/\partial C = (4\kappa^2v^2 - m^2)^{1/2}Y + 2\kappa\langle A\rangle(C - \langle C\rangle) + \ldots$, where the ellipses stand for terms quadratic in fields with zero expectation value. We see that the X field is massless. To compute the masses of the C and Y fields, we first note that, in $|\partial W/\partial C|$, we can absorb the phase of $\langle A\rangle$ into the phase of Y, and so we can replace $\langle A\rangle$ with $|\langle A\rangle|$. Setting $C = \langle C\rangle + (C_1 + iC_2)/\sqrt{2}$ and $Y = (Y_1 + iY_2)/\sqrt{2}$, we find

$$V = 2\kappa^2\langle C\rangle^2 C_1^2 + 2\kappa^2 v^2 C_2^2 + \tfrac{1}{2}\Big|(4\kappa^2v^2 - m^2)^{1/2}(Y_1 + iY_2) + 2\kappa|\langle A\rangle|(C_1 + iC_2)\Big|^2 + \ldots \ , \tag{95.85}$$

where the ellipses stand for cubic and quartic terms. We thus get a mass-squared matrix for C_1 and Y_1 of the form

$$m^2_{C_1,Y_1} = \begin{pmatrix} 4\kappa^2|\langle A\rangle|^2 + 4\kappa^2\langle C\rangle^2 & 2\kappa|\langle A\rangle|\langle C\rangle \\ 2\kappa|\langle A\rangle|\langle C\rangle & 4\kappa^2v^2 - m^2 \end{pmatrix} . \tag{95.86}$$

The mass-squared matrix for C_2 and Y_2 is the same, with $\langle C\rangle \to v$.

The fermion mass matrix is given by $\partial^2 W/\partial A_i \partial A_j$; taking the fields to be X, Y, and C, we find the ψ_X is massless, and that the mass matrix for ψ_Y and ψ_C is

$$m_{\psi_Y,\psi_C} = \begin{pmatrix} 0 & (4\kappa^2v^2 - m^2)^{1/2} \\ (4\kappa^2v^2 - m^2)^{1/2} & 2\kappa\langle A\rangle \end{pmatrix} . \tag{95.87}$$

Since ψ_X is the only massless fermion, it must be the goldstino. To verifiy this, we note that $\langle -F_X^\dagger\rangle = \langle \partial W/\partial X\rangle = (v^2 - m^2/4\kappa^2)^{1/2}$, while $\langle \partial W/\partial Y\rangle = \langle \partial W/\partial C\rangle = 0$.

95.4) a) The kinetic terms for the components of Φ are given by eq. (95.64); to get the kinetic terms for the components of $\bar\Phi$, we take $g \to -g$. The gauge field kinetic terms are given by eq. (95.77). The terms from the superpotential are given by eq. (95.37). Putting it all together, we have

$$\begin{aligned} \mathcal{L} = &-(D^\mu A)^\dagger D_\mu A + i\psi^\dagger\bar\sigma^\mu D_\mu\psi + F^\dagger F + \sqrt{2}e\psi^\dagger\lambda^\dagger A + \sqrt{2}eA^\dagger\lambda\psi - eA^\dagger DA \\ &-(\bar D^\mu \bar A)^\dagger \bar D_\mu \bar A + i\bar\psi^\dagger\bar\sigma^\mu D_\mu\bar\psi + \bar F^\dagger \bar F - \sqrt{2}e\bar\psi^\dagger\lambda^\dagger\bar A - \sqrt{2}e\bar A^\dagger\lambda\bar\psi + e\bar A^\dagger D\bar A \\ &- \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\lambda^\dagger\bar\sigma^\mu\partial_\mu\lambda + \tfrac{1}{2}D^2 \\ &+ m(A\bar F + F\bar A - \psi\bar\psi + A^\dagger\bar F^\dagger + F^\dagger\bar A^\dagger - \psi^\dagger\bar\psi^\dagger) \ , \end{aligned} \tag{95.88}$$

where $D_\mu = \partial_\mu - ieA_\mu$ and $\bar D_\mu = \partial_\mu + ieA_\mu$.

b) Eliminating F and $\bar F$ yields $-m^2(A^\dagger A + \bar A^\dagger \bar A)$. Eliminating D yields $-\frac{1}{2}e^2(A^\dagger A - \bar A^\dagger\bar A)^2$.

95.5) a) Adding $e\xi D$ to $\mathcal{L}$ preserves supersymmetry because an infinitesimal supersymmetry transformation yields a total derivative. If the gauge group was nonabelian, D would carry an adjoint index, and so ξD would not be gauge invariant.

b) The terms involving D in $\mathcal{L}$ are now $\frac{1}{2}D^2 - eD(A^\dagger A - \bar{A}^\dagger \bar{A} - \xi)$, and so eliminating D yields $-\frac{1}{2}e^2(A^\dagger A - \bar{A}^\dagger \bar{A} - \xi)^2$. Eliminating the F and $\bar{F}$ fields yields $-m^2(A^\dagger A + \bar{A}^\dagger \bar{A})$ as before.

c) We have

$$V = m^2(A^\dagger A + \bar{A}^\dagger \bar{A}) + \tfrac{1}{2}e^2(A^\dagger A - \bar{A}^\dagger \bar{A} - \xi)^2 \ . \tag{95.89}$$

Then

$$\begin{aligned} \partial V/\partial A^\dagger &= [m^2 + e^2(A^\dagger A - \bar{A}^\dagger \bar{A} - \xi)]A \ , \\ \partial V/\partial \bar{A}^\dagger &= [m^2 - e^2(A^\dagger A - \bar{A}^\dagger \bar{A} - \xi)]\bar{A} \ . \end{aligned} \tag{95.90}$$

The factors in square brackets sum to $2m^2$, so both cannot vanish. Therefore, either A or $\bar{A}$ (or both) must vanish. If both vanish, we have $V = V_{0,0} = \frac{1}{2}e^2\xi^2$ at the minimum. If we take $\bar{A} = 0$ and $A \neq 0$, then the first square bracket must vanish, which yields $A^\dagger A = \xi - m^2/e^2$; since $A^\dagger A > 0$, this is possible only if $\xi > m^2/e^2$. We then have $V = V_{\emptyset,0} = m^2\xi - \frac{1}{2}m^4/e^2$ at the minimum. Since $V_{\emptyset,0} - V_{0,0} = -\frac{1}{2}(m^2/e^2 - \xi)^2$, the minimum with $\bar{A} \neq 0$ is lower in energy. Finally, if we take $\bar{A} = 0$ and $A \neq 0$, the situation is the same, but with $\xi \to -\xi$. (This is obvious from the form of the potential.) We conclude that A acquires a nonzero VEV if $\xi > m^2/e^2$, and that $\bar{A}$ acquires a nonzero VEV if $\xi < -m^2/e^2$.

To see that supersymmetry is broken, we note that $D = (A^\dagger A - \bar{A}^\dagger \bar{A} - \xi)/e$. Then $\langle D\rangle = -m^2/e$ for $\xi > m^2/e^2$ and $D = +m^2/e$ for $\xi < m^2/e^2$. Also, since $F^\dagger = -m\bar{A}$ and $\bar{F}^\dagger = -mA$, either $\langle F\rangle$ or $\langle \bar{F}\rangle$ is also nonzero. The massless goldstino is then a linear combination of λ and ψ (if $\bar{A} \neq 0$) or $\bar{\psi}$ (if $A \neq 0$).

95.6) a) The net R charge of any term in $\mathcal{L}$ must be zero. Since $\mathcal{L}$ has Yukawa couplings of the form $A^\dagger \psi\lambda$, and $R_\lambda = 1$, we must have $R_A = R_\psi + 1$.

b) The superpotential yields Yukawa couplings of the form $(\partial^2 W/\partial A_i \partial A_j)\psi_i\psi_j$. If W has R charge R_W, then $\partial^2 W/\partial A_i \partial A_j$ has R charge $R_W - R_{A_i} - R_{A_j}$, while $\psi_i\psi_j$ has R charge $R_{\psi_i} + R_{\psi_j}$. Thus $(\partial^2 W/\partial A_i \partial A_j)\psi_i\psi_j$ has R charge $R_W - R_{A_i} - R_{A_j} + R_{\psi_i} + R_{\psi_j} = R_W - 2$, and this must vanish; therefore we must have $R_W = 2$.

c) In SQED, $W = m\bar{A}A$, and so we must assign R charges to A and $\bar{A}$ such that $R_A + R_{\bar{A}} = 2$. Since A and $\bar{A}$ have opposite electric charge, it is most convenient to assign them both R charge $+1$.

96 The Minimal Supersymmetric Standard Model

96.1) In the Standard Model, the Yukawa couplings are of the form $H\ell\bar{e}$, $Hq\bar{d}$, and $H^\dagger q\bar{u}$. If we get the first two from terms in the superpotential of the form $HL\bar{E}$ and $HQ\bar{D}$, we cannot get the third, since the superpotential cannot depend on hermitian conjugates of fields. Thus we need a second Higgs field, $\bar{H}$, with opposite hypercharge.

96.2) a) After solving for the D fields, we have $V_{\text{quartic}} = \frac{1}{2}D^a D^a + \frac{1}{2}D^2$, with $D = \frac{1}{2}g_1(\bar{H}^\dagger\bar{H} - H^\dagger H)$ and $D^a = g_2(H^\dagger T^a H + \bar{H}^\dagger T^a \bar{H})$, where $T^a = \frac{1}{2}\sigma^a$.

b) The only way the potential could be unbounded below is if the quartic terms vanish, so we would need $D = 0$ and $D^a = 0$. Setting $H = \begin{pmatrix} v \\ 0 \end{pmatrix}$ and $\bar{H} = \begin{pmatrix} 0 \\ v \end{pmatrix}$ achieves this. In this case, the mass terms become $(m_1^2 + m_2^2 - 2m_3^2)v^2$, so we must have $m_1^2 + m_2^2 > 2m_3^2$ for the potential to be bounded below.

c) To have symmetry breaking, we need a negative eigenvalue of the mass-squared matrix $\begin{pmatrix} m_1^2 & m_3^2 \\ m_3^2 & m_2^2 \end{pmatrix}$, which requires $(m_3^2)^2 > m_1^2 m_2^2$.

d) One linear combination of H and $\bar{H}$ gets a VEV, and the other does not. The one that gets a VEV produces three Goldstone bosons that are eaten by the gauge fields, and one neutral Higgs boson. The remaining linear combination has one component with unit electric charge, and one component with zero electric charge; each component is a complex scalar field. So in all we have one particle with positive charge, one with negative charge, and three with zero charge.

e) Setting $H = \frac{1}{\sqrt{2}}\begin{pmatrix} v \\ 0 \end{pmatrix}$ and $\bar{H} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \bar{v} \end{pmatrix}$, we find

$$V = \tfrac{1}{2}m_1^2 v^2 + \tfrac{1}{2}m_2^2 \bar{v}^2 - m_3^2 v\bar{v} + \tfrac{1}{32}(g_1^2 + g_2^2)(v^2 - \bar{v}^2)^2 \ . \tag{96.7}$$

Differentiating with respect to v and $\bar{v}$ and setting the results to zero, we find

$$m_1^2 v - m_3^2 \bar{v} + \tfrac{1}{8}(g_1^2 + g_2^2)(v^2 - \bar{v}^2)v = 0 \ , \tag{96.8}$$

$$m_2^2 \bar{v} - m_3^2 v - \tfrac{1}{8}(g_1^2 + g_2^2)(v^2 - \bar{v}^2)\bar{v} = 0 \ . \tag{96.9}$$

If we divide eq. (96.8) by v and eq. (96.9) by $\bar{v}$ and add, we get

$$m_1^2 + m_2^2 - m_3^2\left(\frac{\bar{v}}{v} + \frac{v}{\bar{v}}\right) = 0 \ , \tag{96.10}$$

which immediately yields eq. (96.7).

97 Grand Unification

97.1) With the usual normalization $A(5) = 1$, we have $A(\bar{5}) = -1$. In the notation of problem 70.4, $10 = \mathcal{A}$, and for SU(N), $A(\mathcal{A}) = N-4$. Thus $A(10) = 1$ for SU(5), and $A(\bar{5} \oplus 10) = A(\bar{5}) + A(10) = -1 + 1 = 0$. So the SU(5) model is not anomalous.

There is another, more physical, way to see this. Let Q be the electric charge generator in the $\bar{5} \oplus 10$ representation. We know that all charged fermions can be represented by massive Dirac fields, and so we know that $\operatorname{Tr} Q^3$ must be zero. On the other hand, $\operatorname{Tr} Q^3 \propto A(\bar{5} \oplus 10) d^{QQQ}$, so either $A(\bar{5} \oplus 10) = 0$, or $d^{QQQ} = 0$. To rule out the latter possibility, we compute $\operatorname{Tr} Q^3$ for a single $\bar{5}$; we get $3(+\frac{1}{3})^3 + (-1)^3 \neq 0$. Thus it must be that $A(\bar{5} \oplus 10) = 0$. [This argument is due to R. Cahn, Phys. Lett. B104, 282 (1981).]

97.2) We have

$$\mathcal{L}^{|\Delta B|=1}_{\phi,\text{eff}} = \frac{1}{M_\phi^2}(y\varepsilon^{\alpha\beta\gamma}\bar{d}^\dagger_\alpha \bar{u}^\dagger_\beta)(y\varepsilon^{ij} q_{\alpha i}\ell_j + y''\bar{u}^\dagger_\alpha \bar{e}^\dagger) + \text{h.c.} \,. \tag{97.48}$$

The first term has the same structure as the first term of eq. (97.25) from X exchange, but the second term of eq. (97.48) has a different structure than the second term of eq. (97.25).

97.3) As discussed in problems 88.7 and 89.5, diagrams where a gauge boson connects two fermions with different handedness give no contribution to Z_C in Lorenz gauge. Diagrams where a gauge boson connects fermions of the same handedness yields Z_m in Lorenz gauge, with appropriate replacements of charge/group factors. For Z_{C_1}, the gauge boson must connect ℓ and q, or $\bar{d}^\dagger$ and $\bar{u}^\dagger$, and these contributions add. The appropriate replacements are those shown in eq. (97.44). For SU(N), we have $(T^a_{\rm N})_{\alpha'}{}^\alpha (T^a_{\rm N})_{\beta'}{}^\beta = \frac{1}{2}(\delta_{\alpha'}{}^\beta \delta_{\beta'}{}^\alpha - \frac{1}{N}\delta_{\alpha'}{}^\alpha \delta_{\beta'}{}^\beta)$. Thus $\varepsilon^{\alpha'\beta'\gamma}(T^a_3)_{\alpha'}{}^\alpha (T^a_3)_{\beta'}{}^\beta = \frac{1}{2}(\varepsilon^{\beta\alpha\gamma} - \frac{1}{3}\varepsilon^{\alpha\beta\gamma}) = -\frac{2}{3}\varepsilon^{\alpha\beta\gamma}$ and $\varepsilon^{i'j'}(T^a_2)_{i'}{}^i (T^a_2)_{j'}{}^j = \frac{1}{2}(\varepsilon^{ji} - \frac{1}{2}\varepsilon^{ij}) = -\frac{3}{4}\varepsilon^{ij}$, and so

$$\begin{aligned} Z_{C_1} &= 1 - \tfrac{3}{8\pi^2}(\tfrac{2}{3}g_3^2 + \tfrac{3}{4}g_2^2 + \tfrac{11}{36}g_1^2)\varepsilon^{-1} \\ &= 1 - \tfrac{1}{2\pi}(2\alpha_3 + \tfrac{9}{4}\alpha_2 + \tfrac{11}{12}\alpha_1)\varepsilon^{-1} \end{aligned} \tag{97.49}$$

b) The relevant replacement is

$$\begin{aligned} (-1)(+1)e^2 \to{} & \Big[0 + \varepsilon^{\alpha'\beta'\gamma}(T^a_3)_{\alpha'}{}^\alpha (T^a_3)_{\beta'}{}^\beta / \varepsilon^{\alpha\beta\gamma}\Big] g_3^2 \\ & + \Big[0 + \varepsilon^{i'j'}(T^a_2)_{i'}{}^i (T^a_2)_{j'}{}^j / \varepsilon^{ij}\Big] g_2^2 \\ & + \Big[(+1)(-\tfrac{2}{3}) + (+\tfrac{1}{6})(+\tfrac{1}{6})\Big] g_1^2 \,, \end{aligned} \tag{97.50}$$

and so

$$\begin{aligned} Z_{C_2} &= 1 - \tfrac{3}{8\pi^2}(\tfrac{2}{3}g_3^2 + \tfrac{3}{4}g_2^2 + \tfrac{23}{36}g_1^2)\varepsilon^{-1} \,. \\ &= 1 - \tfrac{1}{2\pi}(2\alpha_3 + \tfrac{9}{4}\alpha_2 + \tfrac{23}{12}\alpha_1)\varepsilon^{-1} \,. \end{aligned} \tag{97.51}$$

c) In Lorenz gauge, we have $Z_2 = 1$ for each fermion lline (at one-loop order). From problem 88.7, we then have that (at one-loop order) γ_i is given by the coefficient of ε^{-1} in Z_{C_i}. Thus,

$$\gamma_1 = -\tfrac{1}{2\pi}(2\alpha_3 + \tfrac{9}{4}\alpha_2 + \tfrac{11}{12}\alpha_1) \,, \tag{97.52}$$

$$\gamma_2 = -\tfrac{1}{2\pi}(2\alpha_3 + \tfrac{9}{4}\alpha_2 + \tfrac{23}{12}\alpha_1) \,. \tag{97.53}$$

d) Note that eq. (97.26) is equivalent to $\mu\, d\alpha_i/d\mu = (b_i/2\pi)\alpha_i^2$, and that, for $n = 3$, we have $b_3 = -7$, $b_2 = -\frac{19}{6}$, and $b_1 = +\frac{41}{6}$. From eq. (88.47), we then have

$$C_1(\mu) = \left[\frac{\alpha_3(\mu)}{\alpha_3(M_{\rm X})}\right]^{2/7} \left[\frac{\alpha_2(\mu)}{\alpha_2(M_{\rm X})}\right]^{27/38} \left[\frac{\alpha_1(\mu)}{\alpha_1(M_{\rm X})}\right]^{-11/82} C_1(M_{\rm X})\ , \tag{97.54}$$

$$C_2(\mu) = \left[\frac{\alpha_3(\mu)}{\alpha_3(M_{\rm X})}\right]^{2/7} \left[\frac{\alpha_2(\mu)}{\alpha_2(M_{\rm X})}\right]^{27/38} \left[\frac{\alpha_1(\mu)}{\alpha_1(M_{\rm X})}\right]^{-23/82} C_2(M_{\rm X})\ . \tag{97.55}$$

We can apply these down to $\mu = M_{\rm Z}$. At that point, SU(2) × U(1) is broken. (We should apply electromagnetic renormalization below this scale, but this is a small effect that we will ignore.) Also, the top quark no longer contributes to b_3, which therefore changes from -7 to $-\frac{23}{3}$. Thus we have, for $\mu < M_{\rm Z}$,

$$C_1(\mu) = \left[\frac{\alpha_3(\mu)}{\alpha_3(M_{\rm Z})}\right]^{6/23} \left[\frac{\alpha_3(M_{\rm Z})}{\alpha_3(M_{\rm X})}\right]^{2/7} \left[\frac{\alpha_2(M_{\rm Z})}{\alpha_2(M_{\rm X})}\right]^{27/38} \left[\frac{\alpha_1(M_{\rm Z})}{\alpha_1(M_{\rm X})}\right]^{-11/82} C_1(M_{\rm X})\ , \tag{97.56}$$

$$C_2(\mu) = \left[\frac{\alpha_3(\mu)}{\alpha_3(M_{\rm Z})}\right]^{6/23} \left[\frac{\alpha_3(M_{\rm Z})}{\alpha_3(M_{\rm X})}\right]^{2/7} \left[\frac{\alpha_2(M_{\rm Z})}{\alpha_2(M_{\rm X})}\right]^{27/38} \left[\frac{\alpha_1(M_{\rm Z})}{\alpha_1(M_{\rm X})}\right]^{-23/82} C_2(M_{\rm X})\ . \tag{97.57}$$

Now we can compute the numerical values, using $\alpha_3(M_{\rm Z}) = 0.1187$, $\alpha(M_{\rm Z}) = 1/127.91$, and (for self-consistency) the SU(5) prediction $\sin^2\theta_{\rm W}(M_{\rm Z}) = 0.207$, as well as $\frac{5}{3}\alpha_1(M_{\rm X}) = \alpha_2(M_{\rm X}) = \alpha_3(M_{\rm X}) = \alpha_5(M_{\rm X}) = 1/41.5$. We also use eq. (97.30) with $M_{\rm X}$ replaced by $M_{\rm Z}$ and $b_3 = -\frac{23}{3}$ to compute $\alpha_3(\mu)$ at $\mu = 2\,{\rm GeV}$, with the result $\alpha_3(2\,{\rm GeV}) = 0.266$. We get

$$C_1(2\,{\rm GeV}) = 2.82\, C_1(M_{\rm X})\ , \tag{97.58}$$

$$C_2(2\,{\rm GeV}) = 2.98\, C_2(M_{\rm X})\ . \tag{97.59}$$

Using $C_1(M_{\rm X}) = C_2(M_{\rm X}) = 4\pi\alpha_5(M_{\rm X})/M_{\rm X}^2$ with $M_{\rm X} = 7{\times}10^{14}$ GeV (the one-loop prediction), we find

$$C_1(2\,{\rm GeV}) = 1.7\times 10^{-30}\,{\rm GeV}^{-2}\ , \tag{97.60}$$

$$C_2(2\,{\rm GeV}) = 1.8\times 10^{-30}\,{\rm GeV}^{-2}\ . \tag{97.61}$$

For more details, see L. F. Abbott and M. B. Wise, Phys. Rev. D22, 2208 (1980).

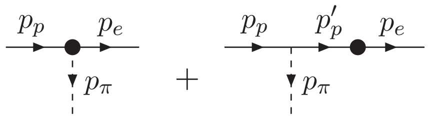
97.4) a) From eq. (83.26), we see that $P_{\rm L}(u\mathcal{N}) = P_{\rm L}N$ and $P_{\rm R}(u^\dagger\mathcal{N}) = P_{\rm R}N$. Then, from eqs. (83.20) and (83.21), we see that (by definition) $P_{\rm L}N$ transforms as $(2,1)$ under ${\rm SU(2)_L}\times{\rm SU(2)_R}$ while $P_{\rm R}N$ transforms as $(1,2)$.

b) After replacing the quark operators by the corresponding hadron operators, eq. (97.46) is simply the translation into Dirac notation of eq. (97.45).

c) We have $\mathcal{N} = \begin{pmatrix} p \\ n \end{pmatrix}$, $u = I + \dfrac{i}{2f_\pi}\begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}$, and $u^\dagger = I - \dfrac{i}{2f_\pi}\begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}$; see eq. (94.29). Thus $(u\mathcal{N})_1 = p + (i/2f_\pi)\pi^0 p + \ldots$ and $(u^\dagger\mathcal{N})_1 = p - (i/2f_\pi)\pi^0 p + \ldots$. Thus we have

$$\mathcal{L}_{\rm eff} = A\,\overline{\mathcal{E}^{\rm C}}(C_1 P_{\rm L} + 2C_2 P_{\rm R})p + i(A/2f_\pi)\pi^0\overline{\mathcal{E}^{\rm C}}(C_1 P_{\rm L} - 2C_2 P_{\rm R})p + \text{h.c.}\,. \tag{97.62}$$

d) The contributing diagrams are



The dot denotes a vertex from $\mathcal{L}_{\rm eff}$; the vertex with no dot in the second diagram is from the usual pion-nucleon interaction, $(g_{\rm A}/2f_\pi)\partial_\mu\pi^0\overline{p}\gamma^\mu\gamma_5 p$, which yields a vertex factor of $(ig_{\rm A}/2f_\pi)(-ip_\pi^\mu)(\gamma_\mu\gamma_5) = (g_{\rm A}/2f_\pi)\not{p}_\pi\gamma_5$. The vertex factor for the dot in the first diagram is $i^2(A/2f_\pi)(C_1P_{\rm L} - 2C_2P_{\rm R})$. The vertex factor for the dot in the second diagram is $iA(C_1P_{\rm L} + 2C_2P_{\rm R})$. Thus we have

$$\text{1st diagram} = -(A/2f_\pi)\overline{u}_e(C_1P_{\rm L} - 2C_2P_{\rm R})u_p \;, \tag{97.63}$$

$$\text{2nd diagram} = +(Ag_{\rm A}/2f_\pi)\overline{u}_e(C_1P_{\rm L} + 2C_2P_{\rm R})\,\frac{-\not{p}_p' + m_p}{p_p'^2 + m_p^2}\,(\not{p}_\pi\gamma_5)u_p \;. \tag{97.64}$$

By momentum conservation $p_p' = p_e$, and the positron is on-shell, $p_e^2 = -m_e^2$, which we neglect compared to m_p^2. Also, we can pull $-\not{p}_e$ to the left, and use $\bar{u}_e\not{p}_e = -m_e\overline{u}_e$, which we can also neglect. Then we use $p_\pi = p_p - p_e$ to get

$$\text{2nd diagram} = +(Ag_{\rm A}/2f_\pi)\overline{u}_e(C_1P_{\rm L} + 2C_2P_{\rm R})\frac{1}{m_p}(\not{p}_p - \not{p}_e)\gamma_5 u_p \;. \tag{97.65}$$

Again we can pull $\not{p}_e$ to the left and replace it with $-m_e$, which we can neglect. Then we use $\not{p}_p\gamma_5 u_p = -\gamma_5\not{p}_p u_p = +\gamma_5 m_p u_p$, followed by $P_{\rm L}\gamma_5 = -P_{\rm L}$ and $P_{\rm R}\gamma_5 = +P_{\rm R}$, to get

$$\text{2nd diagram} = -(Ag_{\rm A}/2f_\pi)\overline{u}_e(C_1P_{\rm L} - 2C_2P_{\rm R})u_p \;. \tag{97.66}$$

From eqs. (97.63) and (97.66), we see that the scattering amplitude $i\mathcal{T}$, given by the sum of the diagrams, is

$$i\mathcal{T} = -\,\frac{A(1{+}g_{\rm A})}{2f_\pi}\,\overline{u}_e(C_1P_{\rm L} - 2C_2P_{\rm R})u_p \;. \tag{97.67}$$

e) Summing over the positron spin and averaging over the proton spin, we have

$$\begin{aligned}
\langle|\mathcal{T}|^2\rangle &= \frac{A^2(1{+}g_{\rm A})^2}{8f_\pi^2}\,\mathrm{Tr}(-\not{p}_p{+}m_p)(C_1P_{\rm R} - 2C_2P_{\rm L})(-\not{p}_e)(C_1P_{\rm L} - 2C_2P_{\rm R}) \\
&= \frac{A^2(1{+}g_{\rm A})^2}{8f_\pi^2}\,\mathrm{Tr}(-\not{p}_p{+}m_p)(C_1P_{\rm R} - 2C_2P_{\rm L})(C_1P_{\rm R} - 2C_2P_{\rm L})(-\not{p}_e) \\
&= \frac{A^2(1{+}g_{\rm A})^2}{8f_\pi^2}\,\mathrm{Tr}(-\not{p}_p{+}m_p)(C_1^2P_{\rm R} + 4C_2^2P_{\rm L})(-\not{p}_e) \\
&= \frac{A^2(1{+}g_{\rm A})^2}{8f_\pi^2}\,(\tfrac{1}{2}C_1^2 + 2C_2^2)(-4p_p{\cdot}p_e) \;.
\end{aligned} \tag{97.68}$$

Now we use $-2p_p{\cdot}p_e = (p_p - p_e)^2 - p_p^2 - p_e^2 = p_\pi^2 - p_p^2 - p_e^2 = -m_\pi^2 + m_p^2 + m_e^2$, and neglecting m_e we have

$$\langle|\mathcal{T}|^2\rangle = \frac{A^2(1{+}g_{\rm A})^2}{8f_\pi^2}(m_p^2 - m_\pi^2)(C_1^2 + 4C_2^2) \;. \tag{97.69}$$

We now have $\Gamma = (|\mathbf{p}_e|/8\pi m_p^2)\langle|\mathcal{T}|^2\rangle$ with $|p_e| = (m_p^2 - m_\pi^2)^{1/2}/2m_p$, and so

$$\Gamma = \frac{A^2(1+g_{\rm A})^2}{128\pi f_\pi^2 m_p^3}(m_p^2 - m_\pi^2)^2(C_1^2 + 4C_2^2)\ . \tag{97.70}$$

Putting in numbers we get $\Gamma = 1.8 \times 10^{-63}\,\text{GeV}$ and $\tau = 1/\Gamma = 1.1 \times 10^{31}\,\text{yr}$. The naive estimate $\Gamma \sim g_5^4 m_p^5/8\pi M_{\rm X}^4$ yields $\Gamma \sim 1.6 \times 10^{-62}\,\text{GeV}$ for $g_5 \sim 0.6$ and $M_{\rm X} = 7 \times 10^{14}\,\text{GeV}$, too large by a factor of 10.

For more details see M. Claudson, M. B. Wise, and L. J. Hall, Nucl. Phys. B 195, 297 (1982); O. Kaymakcalan, L. Chong-Huah, and K. C. Wali, Phys. Rev. D 29, 1962 (1984). For the lattice determination of A (called α in the papers just cited), see N. Tsutsui et al, Phys. Rev. D70, 111501R (2004).

97.5) a) Gluons do not couple to φ. The SU(2) structure of both terms is the same, so the SU(2) contributions to Z_y and $Z_{y'}$ are the same, and hence cancel in the ratio. The U(1) contribution to Z_y from gauge boson lines that connect to φ and to one fermion line is proportional to $Y_\varphi(Y_\ell + Y_{\overline{e}})g_1^2$, while the contribution to $Z_{y'}$ is proportional to $Y_\varphi(Y_q + Y_{\overline{d}})g_1^2$. However, hypercharge conservation requires $Y_\varphi + Y_\ell + Y_{\overline{e}} = 0$ and $Y_\varphi + Y_q + Y_{\overline{d}} = 0$, so these contributions are both proportional to $-Y_\varphi^2 g_1^2$, and hence cancel in the ratio.

b) The argument is the same as in problem 97.3, and the appropriate replacements are $(-1)(+1)e^2 \to (-\frac{1}{2})(+1)g_1^2$ for Z_y, and $(-1)(+1)e^2 \to -C(3)g_3^2 + (+\frac{1}{6})(+\frac{1}{3})g_1^2$ for $Z_{y'}$, where $C(3) = \frac{4}{3}$ is the quadratic Casimir for the fundamental representation of SU(3).

c) We have $y_0 = Z_y Z_\ell^{-1/2} Z_{\overline{e}}^{-1/2} Z_\varphi^{-1/2} y$ and $y_0' = Z_{y'} Z_a^{-1/2} Z_{\overline{d}}^{-1/2} Z_\varphi^{-1/2} y'$. In Lorenz gauge, $Z_{\ell,q,\overline{e},\overline{d}} = 1$ at one-loop order, and so $r_0 = (Z_{y'}/Z_y)r$, with $Z_{y'}/Z_y = 1 - \frac{3}{2\pi}(\frac{4}{3}\alpha_3 - \frac{5}{9}\alpha_1)\varepsilon^{-1}$. The anomalous dimension γ of r is the coefficient of ε^{-1}, thus

$$\gamma = -\tfrac{1}{2\pi}(4\alpha_3 - \tfrac{5}{3}\alpha_1)\ . \tag{97.71}$$

From eq. (88.47), and using $b_3 = -7$ and $b_1 = +\frac{41}{6}$, we have

$$r(M_{\rm Z}) = \left[\frac{\alpha_3(M_{\rm Z})}{\alpha_3(M_{\rm X})}\right]^{4/7}\left[\frac{\alpha_1(M_{\rm Z})}{\alpha_1(M_{\rm X})}\right]^{10/41} r(M_{\rm X})\ . \tag{97.72}$$

Below $M_{\rm Z}$, we neglect the top quark, so that now $b_3 = -\frac{23}{3}$, and also neglect neglect electromagnetic renormalization. The result is

$$r(m_b) = \left[\frac{\alpha_3(m_b)}{\alpha_3(M_{\rm Z})}\right]^{12/23}\left[\frac{\alpha_3(M_{\rm Z})}{\alpha_3(M_{\rm X})}\right]^{4/7}\left[\frac{\alpha_1(M_{\rm Z})}{\alpha_1(M_{\rm X})}\right]^{10/41} r(M_{\rm X}) \tag{97.73}$$

Now we can compute the numerical values, using $\alpha_3(M_{\rm Z}) = 0.1187$, $\alpha(M_{\rm Z}) = 1/127.91$, and (for self-consistency) the SU(5) prediction $\sin^2\theta_{\rm W}(M_{\rm Z}) = 0.207$, as well as $r(M_{\rm X}) = 1$, $\frac{5}{3}\alpha_1(M_{\rm X}) = \alpha_2(M_{\rm X}) = \alpha_3(M_{\rm X}) = \alpha_5(M_{\rm X}) = 1/41.5$. We also use eq. (97.30) with $M_{\rm X}$ replaced by $M_{\rm Z}$ and $b_3 = -\frac{23}{3}$ to compute $\alpha_3(\mu)$ at $\mu = m_b = 4.3\,\text{GeV}$, with the result $\alpha_3(m_b) = 0.213$. We get

$$r(m_b) = 3.07\ . \tag{97.74}$$

We therefore predict that $m_\tau(m_b) = m_b(m_b)/r(m_b) = 1.4\,\text{GeV}$. Since we are neglecting electromagnetic renormalization, we can compare this directly to the physical tau mass, 1.8 GeV; we are off by about 30%. This is close enough to encourage the notion that the basic framework might be right, but far enough off that the specific details must be wrong.